KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM

PROCEEDINGS

VOLUME XL

No. 4

President: J. VAN DER HOEVE
Acting-Secretary: H. R. KRUYT

CONTENTS

WEITZENBÖCK, R.: "Bemerkungen über Trivektoren", p. 312.

JAEGER, F. M., and L. BIJKERK: "Investigations on the Complex Salts of the Racemic and Optically-active Cyclohexanediamines with trivalent Cobaltum and Rhodium". IV. On the Crystallographic Properties of the Optically-active Tri-Cyclohexanediamine-Cobaltic Salts and on Ethylenediamine-Cyclohexanediamine-Cobaltic Salts of this Series", p. 316.

HEUX, J. LE, und A. DE KLEYN: "Untersuchungen an den isolierten Augenmuskeln des Kaninchens während des Lagenystagmus bei akuter Alkoholvergiftung", p. 326.

BRUIN, T. L. DE, J. N. LIER and H. J. VAN DE VLIET: "The ZEEMAN-effect of doubly ionized Cerium, Ce III". (Communicated by Prof. P. ZEEMAN). (With one plate), p. 334.

BRUIN, T. L. DE: "New terms in the spark spectra of Argon, A. II and A. III". (Communicated by Prof. P. ZEEMAN), p. 340.

POPKEN, J.: "Eine arithmetische Eigenschaft gewisser ganzer Funktionen". III. (Communicated by Prof. J. G. VAN DER CORPUT), p. 348.

VISSER, CORNELIS: "On certain infinite sequences". (Communicated by Prof. J. G. VAN DER CORPUT), p. 358.

SWELLENGREBEL, N. H., A. DE BUCK and H. KRAAN: "Further investigations on "healthy" human carriers of Plasmodium vivax in North-Holland"). (Communicated by Prof. W. A. P. SCHÜFFNER), p. 368.

Mathematics. — *Bemerkungen über Trivektoren.* Von R. WEITZENBÖCK.

(Communicated at the meeting of March 20, 1937).

§ 1.

Eine Ebene im G_n ($=(n-1)$-dimensionaler linearer Raum), die durch drei, nicht auf einer Geraden liegenden Punkte x, y und z gegeben ist, hat die $\binom{n}{3}$ homogenen Koordinaten $p_{ikl} = (x\,y\,z)_{ikl}$; es sind dies die dreireihigen Determinanten, die sich aus der Matrix

$$\begin{Vmatrix} x_1 & x_2 & . & . & . & . & x_n \\ y_1 & y_2 & . & . & . & . & y_n \\ z_1 & z_2 & . & . & . & . & z_n \end{Vmatrix}$$

herausgreifen lassen. Die algebraische Abhängigkeit dieser p_{ikl} wird durch die quadratischen p-Relationen ausgedrückt, die sich durch

$$p_{[ikl}\, p_{mn]r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

zusammenfassend darstellen lassen. Dabei sind alle Indizes willkürlich und die Klammer $[iklmn]$ bedeutet, dass über die in ihr enthaltenen fünf Indizes alterniert wird.

Aequivalent mit diesen p-Relationen sind die Gleichungen [1]), die sich aus

$$p_{123}^2 \cdot p_{ikl} = \begin{vmatrix} p_{i23} & p_{i31} & p_{i12} \\ p_{k23} & p_{k31} & p_{k12} \\ p_{l23} & p_{l31} & p_{l12} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ergeben und mit deren Hilfe sich alle p_{ikl} rational durch $p_{123} \neq 0$ und die

$$p_{i23}\ ,\ p_{i31}\ ,\ p_{i12} \qquad (i = 4, 5, \ldots, n)$$

ausdrücken lassen. Setzen wir dann $p_{123} = 1$ und

$$p_{i23} = a_i\ ,\ p_{i31} = b_i\ ,\ p_{i12} = c_i \quad (i = 4, 5, \ldots, n) \quad . \quad . \quad . \quad (3)$$

so haben wir aus (2)

$$\left.\begin{array}{l} p_{1ik} = (bc)_{ik} \ = b_i\, c_k - b_k\, c_i \\ p_{2ik} = (ca)_{ik} \ = c_i\, a_k - c_k\, a_i \\ p_{3ik} = (ab)_{ik} \ = a_i\, b_k - a_k\, b_i \\ p_{ikl} \ = (abc)_{ikl} = \begin{vmatrix} a_i & a_k & a_l \\ b_i & b_k & b_l \\ c_i & c_k & c_l \end{vmatrix} \end{array} \quad (i, k, l = 4, 5, \ldots, n) \right\} \quad . \quad . \quad . \quad (4)$$

---

[1]) Vgl. z. B. TH. VAHLEN, Crelle **112** (1893), S. 306—310.

Durch (3) und (4) sind also alle p_{ikl} als ganze rationale Funktionen der $3(n-3)$ Parameter

$$a_i \; , \; b_k \; , \; c_l \qquad (i, k, l = 4, 5, \ldots, n)$$

dargestellt.

Nimmt man die p_{ikl} als homogene Punktkoordinaten in einem Raume $\mathfrak{R}_N$ von $N = \binom{n}{3} - 1$ Dimensionen, so entspricht jedem alternierenden Tensor a_{ikl} (= linearer G_{n-3}-Komplex im G_n) ein Punkt $\mathfrak{A}$ des $\mathfrak{R}_N$. Jeder Ebene p_{ikl} ist ein Punkt $\mathfrak{P}$ zugeordnet und alle Punkte $\mathfrak{P}$ bilden im $\mathfrak{R}_N$ die GRASSMANN'*sche Mannigfaltigkeit* $\mathfrak{M}^k_{3(n-3)}$ von $3(n-3)$ Dimensionen und vom Grade

$$k = \frac{1!\,2!\,(3(n-3))!}{(n-1)!\,(n-2)!\,(n-3)!}.$$

Sie ist rational und wird mit Hilfe der $3(n-3)$ Parameter a_i, b_k und c_l $(i, k, l = 4, 5, \ldots, n)$ durch die Gleichungen (3) und (4) dargestellt (wobei $p_{ikl} = 1$ gesetzt ist).

§ 2.

Wenn ein linearer G_{n-3}-Komplex a_{ikl} als Summe von linear-unabhängigen Ebenen darstellbar ist, so hat man

$$a_{ikl} = \lambda_1 \overset{(1)}{p}_{ikl} + \lambda_2 \overset{(2)}{p}_{ikl} + \ldots + \lambda_s \overset{(s)}{p}_{ikl} \quad \ldots \ldots \quad (5)$$

oder, im Raume $\mathfrak{R}_N$

$$\mathfrak{A} = \lambda_1 \mathfrak{P}^{(1)} + \lambda_2 \mathfrak{P}^{(2)} + \ldots + \lambda_s \mathfrak{P}^{(s)}.$$

Die s Punkte $\mathfrak{P}^{(\nu)}$ bestimmen im $\mathfrak{R}_N$ ein Sekantengebiet $\mathfrak{G}_s$ der GRASSMANN'schen Mannigfaltigkeit $\mathfrak{M}^k_{3(n-3)}$ und die Frage nach einem minimalen s kann also wie folgt eingekleidet werden: wie viel linear-unabhängige Punkte $\mathfrak{P}$ auf $\mathfrak{M}^k_{3(n-3)}$ muss man mindestens wählen, damit der sie verbindende lineare Raum einen willkürlichen Punkt $\mathfrak{A}$ des $\mathfrak{R}_N$ enthält?

Denkt man sich die s Punkte $\mathfrak{P}^{(\nu)}$ durch die $3s(n-3)$ Parameter

$$a_i^{(\nu)} \; , \; b_i^{(\nu)} \; , \; c_i^{(\nu)} \qquad (i = 4, 5, \ldots, n;\; \nu = 1, 2, \ldots, s) \quad \ldots \quad (6)$$

festgehalten, so lassen sich die Gleichungen (5) folgendermassen schreiben

$$\left.\begin{aligned}
a_{123} &= \lambda_1 + \lambda_2 + \ldots + \lambda_s \\
a_{23i} &= \lambda_1 a_i^{(1)} + \lambda_2 a_i^{(2)} + \ldots + \lambda_s a_i^{(s)} \\
a_{31i} &= \lambda_1 b_i^{(1)} + \ldots \\
a_{12i} &= \lambda_1 c_i^{(1)} + \ldots \\
a_{1ik} &= \lambda_1 (b^{(1)} c^{(1)})_{ik} + \lambda_2 (b^{(2)} c^{(2)})_{ik} + \ldots + \lambda_s (b^{(s)} c^{(s)})_{ik} \\
a_{2ik} &= \lambda_1 (c^{(1)} a^{(1)})_{ik} + \ldots \\
a_{3ik} &= \lambda_1 (a^{(1)} b^{(1)})_{ik} + \ldots \\
a_{ikl} &= \lambda_1 (a^{(1)} b^{(1)} c^{(1)})_{ikl} + \ldots + \lambda_s (a^{(s)} b^{(s)} c^{(s)})_{ikl}
\end{aligned}\right\} \quad \ldots \quad (7)$$

Algebraisch formuliert ergibt sich dann die Aufgabe: Gegeben sind die $\binom{n}{3}$ willkürlichen Grössen a_{ikl} und die Gleichungen (7). Aus diesen sind erstens die s Grössen λ_ν und zweitens die $3\,s\,(n-3)$ Parameter (6) zu berechnen. Was ist der kleinste Wert s_n von s, bei dem diese Berechnung stets möglich ist?

Dass man hier mit einer einfachen Konstantenabzählung nicht auskommt, ist leicht zu zeigen. Wir haben $\binom{n}{3}$ Gleichungen (7) und $s+3s\,(n-3)$ Unbekannte, also

$$s+3s\,(n-3) \geqq \binom{n}{3}, \text{ d. h.}$$

$$s_n \geqq \frac{n\,(n-1)\,(n-2)}{6\,(3n-8)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

Für grosse Werte von n wird also das minimale s_n von der Grössenordnung $\frac{n^2}{18}$. Ich habe seinerzeit bewiesen [1]), dass man stets mit

$$s_n \leqq \left[\left(\frac{n-3}{2}\right)^2+1\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

auskommt und dies gibt für grosse Werte von n die Ordnung $\frac{n^2}{4}$, sodass ein Spielraum der Grösse $\frac{7}{36}\,n^2$ offen bleibt. Für die niedrigeren Werte von n geben (9) und (8) die Anzahlen

$n=$	5	6	7	8	9	10	11	12
$s_n \leqq$	2	3	5	7	10	13	17	21
$s_n \geqq$	2	2	3	4	5	6	7	8

Bei $n=5$, 6, 7, 8 und 9 stimmen die Zahlen der zweiten Reihe mit den wirklichen Minimalzahlen überein [2]).

§ 3.

Ich skizziere hier noch einen einfachen Beweis von (9). Es handelt sich darum, eine Summe

$$a_{ikl} = p^{(1)}_{ikl} + p^{(2)}_{ikl} + \ldots$$

---

[1]) Diese Proceed. **32** (1929), p. 248—250.

[2]) Für $n=6$: O. LANDSBERG, Dissertation Breslau (1889); für $n=7$: W. REICHEL Dissertation Greifswald (1907); für $n=8$: W. KÄMMERER, Dissertation GIESSEN (1927) und J. A. SCHOUTEN, Rendiconti di Palermo **55** (1931), p. 137—156; für $n=9$: G. GOUREWITSCH, C. R. Moskau (Mai 1935), p. 353—356.

von willkürlich vielen Ebenen $p_{ikl}^{(\nu)}$ im Gebiete G_n so zu reduzieren, dass rechter Hand möglichst wenig Komponenten stehen.

Man kann annehmen, dass keine zwei dieser Ebenen eine Gerade gemein haben.

Wir legen durch möglichst viele der Ebenen $p_{ikl}^{(\nu)}$ ein Gebiet $G_{n-1}(v')$ und es seien

$$p_{ikl}^{(1)}, p_{ikl}^{(2)}, \dots, p_{ikl}^{(h)} \quad \dots\dots\dots \quad (10)$$

alle diejenigen Ebenen $p_{ikl}^{(\nu)}$, die nicht im $G_{n-1}(v')$ liegen, die also dieses Gebiet nach den h Geraden

$$q_{ik}^{(1)}, q_{ik}^{(2)}, \dots, q_{ik}^{(h)} \quad \dots\dots\dots \quad (11)$$

schneiden, sodass, wenn $y^{(\nu)}$ einen Punkt der Ebene $p_{ikl}^{(\nu)}$ ausserhalb des $G_{n-1}(v')$ bedeutet, jede der Ebenen (10) durch die Verbindung von $y^{(\nu)}$ mit der Geraden $q_{ik}^{(\nu)}$ zustande kommt:

$$p_{ikl}^{(\nu)} = [p_{ik}^{(\nu)} y^{(\nu)}].$$

Sind nun $y^{(\nu)}$ und $y^{(\mu)}$ zwei verschiedene Punkte, so schneidet ihre Verbindungslinie das Gebiet $G_{n-1}(v')$ in einem Punkte z, sodass man z. B. $y^{(\mu)} = y^{(\nu)} + \lambda . z$ setzen darf. Dies gibt dann für die Ebene $p_{ikl}^{(\mu)}$ die Darstellung

$$p_{ikl}^{(\mu)} = [q_{ik}^{(\mu)} y^{(\mu)}] = [q_{ik}^{(\mu)} y^{(\nu)}] + \lambda [q_{ik}^{(\mu)} z],$$

d. h. die Ebene $p_{ikl}^{(\mu)}$ erscheint dargestellt als Summe zweier anderer, von denen die eine auch durch den Punkt $y^{(\nu)}$ geht und die andere im Gebiete v' ligt. Man erhält also an Stelle der h Ebenen (10) ebensoviele, durch einen ausserhalb des Gebietes $G_{n-1}(v')$ liegenden Punkt y gehende Ebenen und weitere, in $G_{n-1}(v')$ liegende Ebenen. Die Gesamtzahl des letzteren ist höchstens s_{n-1}. Die Summe der Ebenen (10) kann dann dargestellt werden durch das äussere Produkt von y mit der Summe der Geraden (11). Diese letzteren liegen alle im Gebiete $G_{n-1}(v')$ und ihre Summe kann daher auf die von höchstens $\left[\frac{n-1}{2}\right]$ Geraden r_{ik} reduziert werden. Es lässt sich dann weiter zeigen, dass eine dieser Geraden r_{ik} noch in einer der in v' liegenden Ebenen p_{ikl} gewählt werden kann, sodass sich die Anzahl der Ebenen um Eins erniedrigt. Wir haben so

$$s_n \leqq s_{n-1} + \left[\frac{n-1}{2}\right] - 1,$$

woraus man (9) erhält.

Chemistry. — *Investigations on the Complex Salts of the Racemic and Optically-active Cyclohexanediamines with trivalent Cobaltum and Rhodium.* **IV.** *On the Crystallographic Properties of the Optically-active Tri-Cyclohexanediamine-Cobaltic Salts and on Ethylene-diamine-Cyclohexanediamine-Cobaltic Salts of this Series.* By F. M. JAEGER and L. BIJKERK.

(Communicated at the meeting of March 20, 1937).

§ 1. In continuation of our study[1] on the optically active *tri-cyclohexanediamine-cobaltic* salts, in this paper we once more wish to draw attention to some remarkable differences of the properties of the corresponding *cyclopentane-* and *cyclohexanediamine-cobaltic* salts produced under comparable circumstances.

In the first place, these differences reveal themselves with respect to the crystalforms of these two classes of complex salts. A direct comparison of the crystalforms of the *chlorides* ($+ 4H_2O$) of both series can make this clear.

Whilst the *tricyclopentanediamine-cobaltic-chloride* $+ 3H_2O$ is *hexagonal*, but never shows a hemimorphic development, and the salt $+ 4H_2O$ is *cubic-pentagone-icositetrahedral*, the *tri-l-cyclohexanediamine-cobaltic chloride* $+ 4H_2O$ crystallizes in the *hexagonal-pyramidal* class, with $a : c = 1 : 2{,}730$. It forms lustrous, reddish-orange crystals of typically hemimorphic habitus, (Fig. 1), with the faces: $c' = \{000\bar{1}\}$, very lustrous, predominant and yielding excellent reflections; $o = \{10\bar{1}3\}$, well developed, giving better reflections than $\omega = \{10\bar{1}\bar{3}\}$, which is much narrower. The symmetry here indicated was corroborated by means of X-ray-spectrograms: the very sharp LAUE-patterns, obtained by rays parallel to the c-axis, only possess *a single hexagonal axis*, but *no* planes of symmetry (Fig. 2). From oscillation-spectrograms, the dimensions of the elementary cell were determined to be $a_0 = 12{,}2$ Å. and $c_0 = 33{,}3$ Å. The fundamental grating is the simple hexagonal one; the structure has

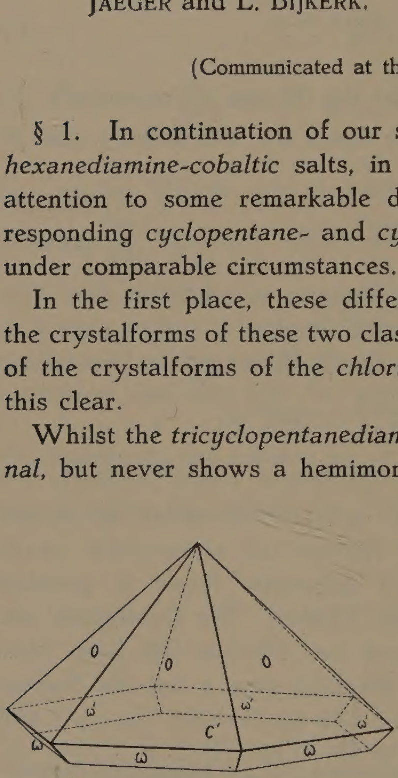


Fig. 1. *The Crystalform of* $D\text{-}\{Co(l\text{-}Chxn)_3\}Cl_3 + 4H_2O$.

---

[1] F. M. JAEGER and L. BIJKERK, these Proceedings, **40**, 116, 246 (1937).

no other symmetry-elements than a hexagonal screw-axis: on the oscillation-spectrograms only the basal reflections (0006) and (000.12) appear to be present. The very large cell contains 6 molecules of the salt; the space-group is either C_6^2 or C_6^3.

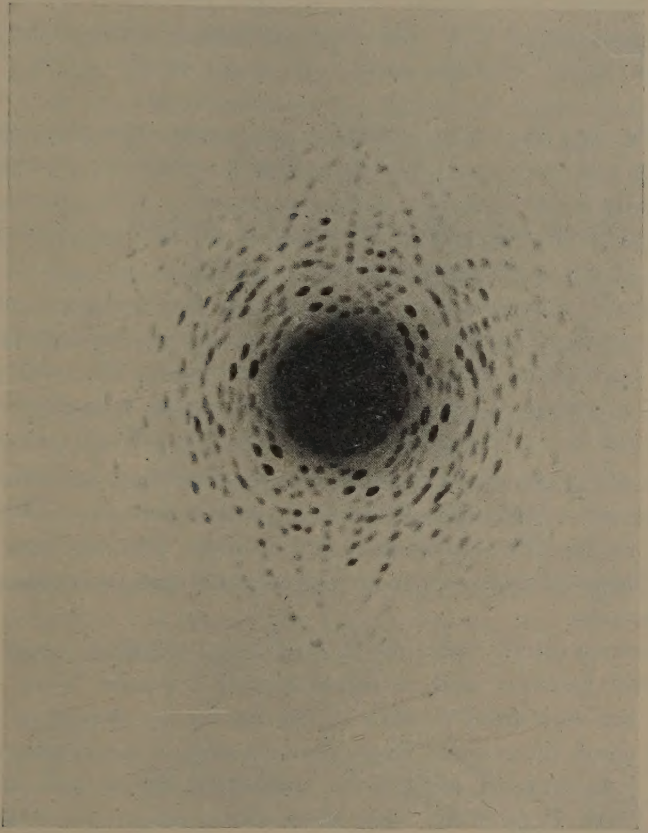


Fig. 2. LAUE-*diagram of* $\{Co(l\text{-}Chxn)_3\}\,Cl_3 + 4H_2O$ *on* $\{0001\}$.

Angular Values:	*Observed*:	*Calculated*:
$o:o = (10\bar{1}3):(01\bar{1}3) =$	*42° 28′	—
$o:o = (01\bar{1}3):(0\bar{1}13) =$	92 43	92° 50′
$o:\omega = (10\bar{1}3):(10\bar{1}\bar{3}) =$	87 17	87 10
$c:\omega = (000\bar{1}):(10\bar{1}\bar{3}) =$	46 26½	46 25

The crystals are completely isomorphous with those of the corresponding *rhodium* salt previously described [1]). They are optically uniaxial, negative and show a dextrorotation of the plane of polarisation. The specific weight is: 1,356 at 18° C.

Analysis of these crystals yielded: 14,42—14,51 % N; 12,38 % H_2O; calculated for a salt with $4H_2O$: 14,51 % N and 12,42 % H_2O. The corresponding salt, obtained by the fission of $\{Co(r\text{-}Chxn)_3\}Cl_3 + 1H_2O$ by means of the *chloro-d-tartrates*, had exactly the same crystalform, but proved to contain only $3H_2O$: 9,74 % H_2O and 14,58 % N; calculated: 9,62 % H_2O and 14,96 % N. The angle: $(0001) : (10\bar{1}3)$ of the *apparently bi*pyramidal crystals (by twinning), was about 46°22′ ± 0°5.

---

[1]) F. M. JAEGER and L. BIJKERK, these Proceedings, **40**, 116 (1937).

§ 2. The optically active *chlorides* are appreciably more soluble than the inactive one, the corresponding *nitrates, chlorates, perchlorates* and *iodides* hardly more than the analogous inactive salts. Several of them were prepared from the *chloride* by means of the appropiate *silver* salts or, in the case of the iodide, by precipitation with NaI.

The optically active *iodide* with 3 mol. of the *l*-base and crystallizing with 2 molecules of water is much more soluble in boiling than in cold water and crystallizes from the hot aqueous solution on cooling in cubic, doubtlessly *pentagone-icositetrahedral,* small, optically isotropous crystals of a blood-red or an orange-red colour. They usually are limited by the octahedron $\{1\bar{1}1\}$, the corners of which are sometimes truncated by small faces of the cube $\{100\}$. The crystals show circular polarisation (dextrogyratory), as Dr. TERPSTRA was able to state, but they are *not* piezo-electrical. These facts also are in agreement with their pentagone-icositetrahedral symmetry.

The dextrogyratory salt: $D\text{-}\{Co(l\text{-}Chxn)_3\}(NO_3)_3 + 3H_2O$ forms small, highly lustrous and orange-coloured crystals. Very beautifully developed, but very small crystals of the nitrate are deposited from a hot solution in dilute nitric acid. Their analysis proved the presence of 3 mol. of water: 7,46 % H_2O; 19,68 % N; calculated for a salt with $3H_2O$: 8,4 % H_2O and 19,66 % N. They show the combination of a hexagonal prism with either an apparent hexagonal bipyramid or with two rhombohedra of opposite signs. Perhaps they are trigonal-trapezohedral, like *quartz*. The axial ratio (hexagonal axes) is: $a : c = 1 : 0{,}7752$.

Forms observed: $m = \{10\bar{1}0\}$ and $p = \{10\bar{1}1\}$ (or a combination of a + and — rhombohedron), both excellently reflecting. By means of the theodolite-goniometer, the angular values:

	Observed:	*Calculated*:
$m : m = (10\bar{1}0) : (01\bar{1}0) =$	*60°	—
$m : p = (10\bar{1}0) : (10\bar{1}1) =$	*48 10′	—
$m : p = (01\bar{1}0) : (\bar{1}101) =$	70 29	70° 31′

were measured, from which the axial ratio was calculated.

The exceptionally close analogy between these dextrogyratory crystals and those of the inactive compounds formerly described 1) also reveals itself in the external aspect, although the latter is only produced by a twinning parallel to $\{0001\}$.

---

1) F. M. JAEGER and L. BIJKERK, these Proceedings, **40**, 116, 246 (1937).

§ 3. The salt $\left\{Co\begin{matrix}(Ene)_2\\(d\text{-}Chxn)\end{matrix}\right\}Cl_3 + 1H_2O$ was prepared either from $\left\{\begin{matrix}(Ene)_2\\Co(H_2O)\\Cl\end{matrix}\right\}_2(SO_4)_3 + 1\frac{1}{2}H_2O$ or from $\left\{Co\begin{matrix}(Ene)_2\\Cl_2\end{matrix}\right\}Cl$ by boiling it with 1 equivalent of *d-cyclohexanediamine* in absolute alcoholic solution; gradually small quantities of water are added during the heating, but an excess of the base must be avoided, because in that case simultaneously some $\left\{Co\begin{matrix}(Ene)\\(d\text{-}Chxn)_2\end{matrix}\right\}Cl_3$ appears to be formed. Subsequently the solution is evaporated on the water-bath, the residue dissolved in as little warm water as possible, and the solution obtained left at room-temperature for crystallization. Soon thin parallelogram-shaped tables or occasionally thicker ones are deposited, which contain 3,29 % H_2O (calculated for the salt with $1H_2O$: 4,3 %), whilst the waterfree substance contains 20,85 % N, whilst 21,05 % N being the theoretically expected value.

The salt obtained is *rhombic-bisphenoidal*, with the axial ratio: $a \,.\, b : c =$ 0.663 : 1 : 0.867 and the forms: $m = \{110\}$, broad and very lustrous; $c = \{001\}$, predominant, yielding sharp, but often multiple reflections; $o = \{111\}$, well developed, lustrous, also giving multiple images; $\omega = \{1\bar{1}1\}$, only rarely present in the thinner plates and very narrow, but well measurable. The habitus is flattened parallel to {001} (Fig. 3).

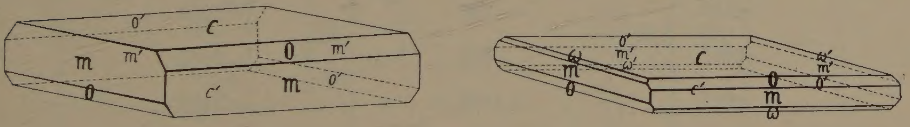


Fig. 3 *Crystalform of* $\left\{Co\begin{matrix}(Ene)_2\\(d\text{-}Chxn)\end{matrix}\right\}Cl_3 + 1\,H_2O.$

Angular Values:	*Observed*:	*Calculated*:
$m : m = (110) : (1\bar{1}0) =$	*67° 4′	—
$c : o = (001) : (111) =$	*57 30	—
$m : o = (110) : (111) =$	32 30	32° 30′
$m : o = (1\bar{1}0(: (1\bar{1}1) =$	71 1	70 49

Perhaps an imperfect cleavability occurs parallel to {001}.

Optically biaxial; the plane of the optical axes is {010}, with the c-axis as first bissectrix and an apparently rather large optical angle.

The rotatory dispersion of this salt was measured; the data obtained are collected in Table I.

The molecular rotations are greater than those of the *tri-d-cyclohexanediamine* salt, but the curve of the rotatory dispersion has a similar shape.

TABLE I.

Rotatory Dispersion of $\left\{ Co \begin{matrix} (Ene)_2 \\ (d\text{-}Chxn) \end{matrix} \right\} Cl_3 + 1\ H_2O$

Wavelength λ in A.U.:	Specific Rotation [α]:	Molecular Rotation [M] . 10^{-2}:
6980	— 37°	— 154°
6730	— 34	— 142
6480	— 31	— 129
6262	— 26.5	— 110
6074	— 12	— 50
5893	+ 19.5	+ 80
5753	+ 77	+ 320
5592	+ 170	+ 708
5463	+ 320	+1333
5340	+ 460	+1917
5224	+ 500 *M*	+2083 *M*
5126	+ 470	+1958
5036	+ 190	+ 792
4950	— 290	—1208
4861	— 760	—3165
4793	—1490 *m*	—8105 *m*
4724	—1200	—5000
4658	—1110	—4625
4596	—1080	—4500
4537	—1030	—4292
4483	— 910	—3792

§ 4. In the mother-liquor just described a crop of prismatic crystals was found, which proved to consist of the salt: $\left\{ Co \begin{matrix} (Ene) \\ (d\text{-}Chxn)_2 \end{matrix} \right\} Cl_3 + 3\frac{1}{2}H_2O$. The analysis yielded a not quite constant watercontent of 11,8—12,2 % H_2O (theoretically, for $4H_2O$: 13,7 %) and 16,13—16,55 % N (theoretically: 16,26 % N); the anhydrous salt gave: 23,31 % Cl, calculated: 23,48 % Cl.

By boiling an aqueous solution of $\left\{ Co \begin{matrix} (d\text{-}Chxn)_2 \\ Cl_2 \end{matrix} \right\} Cl$ with 1 equivalent of *ethylenediamine*, no good results were obtained, because under these conditions evidently a fission of the product formed into the *tricyclohexanediamine-* and the *triethylenediamine* salts occurs.

Better results were obtained when the experiment was repeated by adding *two* equivalents of the base to the *praseo*-salt. In this case a yield of the desired salt was obtained amounting to about half the expected quantity, and about the same yield of the *diethylenediamine-cyclohexanediamine*-compound. The two salts can be separated by fractional crystallizations and identified by their individual crystalforms, because first $\left\{Co\begin{matrix}(Ene)\\(d\text{-}Chxn)_2\end{matrix}\right\}Cl_3$ is deposited from the solution and subsequently the other salt.

The salt $\left\{Co\begin{matrix}(Ene)\\(d\text{-}Chxn)_2\end{matrix}\right\}Cl_3 + 3\frac{1}{2}H_2O$ crystallizes in *rhombic* needles or thicker crystals; most probably they have *rhombic-bisphenoidal* symmetry, although this could neither be corroborated by piezo-electrical phenomena, nor by the study of the habitus of the crystals (Fig. 4).

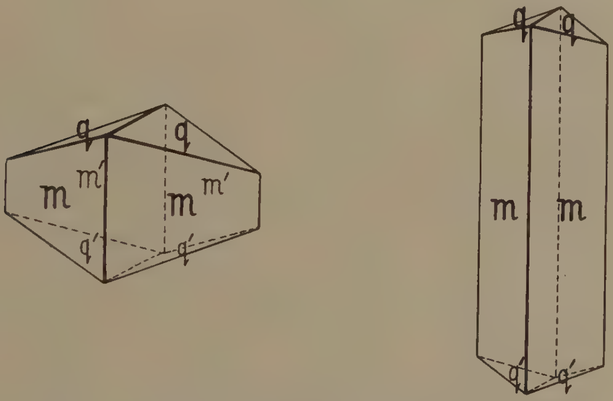


Fig. 4. *Crystalform of* $\left\{Co\begin{matrix}(Ene)\\(d\text{-}Chxn)_2\end{matrix}\right\}Cl_3 + 3\,H_2O$.

The crystals are either apparently octahedrally shaped individuals or long needles; in both cases only the forms $m = \{110\}$ and $q = \{011\}$ are present; the faces of both forms are highly lustrous and the crystals show very constant angular values.

Angular Values:	*Observed:*	*Calculated:*
$m : m = (110) : (\bar{1}10) =$	*70° 4′	—
$q : q = (011) : (0\bar{1}1) =$	*42 7	—
$m : q = (110) : (011) =$	78 7	78° $5\frac{3}{4}$′

No distinct cleavability was observed.

Optically biaxial; the plane of the optical axes is probably {100}.

The rotatory dispersion of this salt was measured in a 0,1 % solution; the data obtained are collected in the following table:

TABLE II.

Rotatory Dispersion of $\left\{ Co \begin{matrix} (Ene) \\ (d\text{-}Chxn)_2 \end{matrix} \right\} Cl_3 + 3^1/_2 H_2O$

Wavelength λ in A.U.:	Specific Rotation $[\alpha]$:	Molecular Rotation $[M] . 10^{-2}$:
6980	— 55°	— 284°
6730	— 53	— 274
6480	— 50	— 258
6262	— 50	— 258
6074	— 50	— 258
5893	— 20	— 103
5753	+ 20	+ 103
5592	+ 120	+ 620
5463	+ 230	+1189
5340	+ 350	+1809
5224	+ 390 *M*	+2015 *M*
5126	+ 259	+1340
5036	+ 30	+ 155
4950	— 670	—3462
4861	—1026	—5300
4793	—1560 *m*	—8060 *m*
4724	—1480	—7646
4658	—1370	—7080
4596	—1290	—6665
4537	—1180	—6097
4483	—1100	—5683

These data, together with those of the previous salt, have already been graphically represented in Fig. 8 of our former paper 1) and the curves obtained have been compared with that of $L\text{-}\{Co(d\text{-}Chxn)_3\}Cl_3$, as well as with that of $D\text{-}\{Co(Ene)_3\}Cl_3$. The analogies of the four curves at once become evident from this figure, as well as the shifts of the different *zero-*

---

1) F. M. JAEGER and L. BIJKERK, these Proceedings, **40**, 257 (1937).

points of the *L*-salts on gradually introducing more molecules of *d-cyclohexanediamine* into the complex ions, instead of the molecules of *ethylenediamine* in the *triethylenediamine*-ion: evidently there is present an *almost pure superposition* of the *levo*gyratory influence of the *d-diamine* molecules introduced upon the rotation of the resolved complex ion itself, so that the rotation in the *red* part of the spectrum becomes *negative*, although the character of the dispersion of the $D\text{-}\{Co(Ene)_3\}^{\cdots}$-ion has actually been preserved.

§ 5. By the interaction of 1 molecule of *ethylenediamine* and 1 molecule of the *praseo*-salt: $\left\{Co\begin{matrix}(r\text{-}Chxn)_2\\ Cl_2\end{matrix}\right\}Cl$, in the same way one can try to prepare the corresponding *dicyclohexanediamine-ethylenediamine-cobaltic chloride*: $\left\{Co\begin{matrix}(r\text{-}Chxn)_2\\ (Ene)\end{matrix}\right\}Cl_3$. The *chloride* obtained is very soluble, but crystallizes from its hot, saturated solutions, in the form of fine, lustrous, brown-red needles which, however, *proved* to be $\{Co(r\text{-}Chxn)_3\}Cl_3$. Evidently, therefore, the racemic salt cannot be prepared in this way.

In the same way the corresponding *diethylenediamine-r-cyclohexanediamine-cobaltic* salt was prepared from the red *diethylenediamine-chloro-aquo-cobaltic sulphate* ($+ 1{,}5\, H_2O$) by dissolving 32 grammes of the latter in 150 cm³ water and, after the addition of 12 grammes of *r-cyclohexanediamine*, by boiling the solution at a reflux-cooler during several hours. On evaporation of the reddish-brown solution to a small volume on the water-bath, a chocolate-brown, very soluble salt is obtained, which is dissolved into a small quantity of cold water and, after heating the solution with *norite*, is precipitated in the form of the orange coloured *iodide* by an excess of *NaI*. The orange precipitate is filtered off, washed with water and alcohol, dissolved in as little boiling water as possible, the solution filtered off from a small quantity of cobaltous oxide and left for crystallization.

The *NaI*-solution still contains some of the impure iodide, which can be obtained from it and purified in the same way; the total yield of the salt obtained is practically the theoretical one.

The *iodide* crystallizes either in the shape of fine, red-orange coloured needles or in octahedrally shaped, small, rhombic crystals, which contain 1—1,5 mol. water of crystallization. The salt can in the same way also be prepared from the *praseo*-salt: $\left\{Co\begin{matrix}(Ene)_2\\ Cl_2\end{matrix}\right\}Cl$.

Analysis: 54,95 % *I*; 8,52 % *Co*; 12,13 % *N* and about 3,7 % H_2O; calculated for $\left\{Co\begin{matrix}(Ene)_2\\ (r\text{-}Chxn)\end{matrix}\right\}I_3 + 1H_2O$: 55,06 % *I*; 8,53 % *Co*; 12,14 % *N* and 2,60 % H_2O.

The crystals are *rhombic-bipyramidal* (Fig. 5), with the axial ratio:

$$a : b : c = 0{,}85 : 1 : 1{,}66.$$

Often the limiting planes are curved, but from oscillations-spectrograms round the three axes the dimensions of the primary cell were found to be:

$$a_0 = 11{,}4\ \text{Å};\ b_0 = 13{,}4\ \text{Å};\ c_0 = 22{,}3\ \text{Å},$$

from which the axial ratio just mentioned can be deduced.

Angular Values:

$$o:o = (111):(1\bar{1}1) = 74^\circ\ 12\tfrac{1}{2}'$$
$$o:o = (111):(11\bar{1}) = 42\ \ 37\tfrac{1}{2}$$
$$o:o = (111):(\bar{1}11) = 90\ \ 26$$

Sometimes the crystals have an apparently hemimorphic shape, but evidently they are holohedral. The crystalform, although analogous to that of the corresponding *triethylene-cobaltic iodide* ($+ 1H_2O$), certainly is different from the latter [1]).

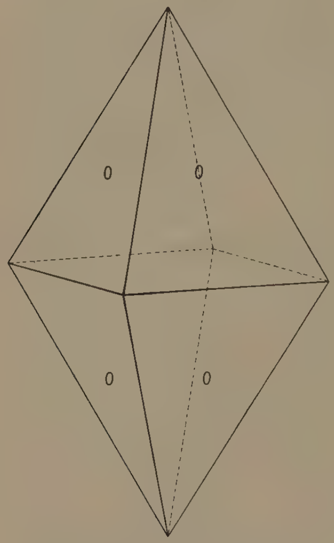


Fig. 5. *Crystalform of* $\left\{Co\begin{matrix}(Eine_2)\\(r\text{-}Chxn)\end{matrix}\right\} I_3 + 1\,H_2O.$

§ 6. Experiments were also made with the purpose of resolving the salts derived from the racemic base into their optically active antipodes by means of the *chloro-d-tartrates*. In the case of $\left\{Co\begin{matrix}(Ene)\\(r\text{-}Chxn)_2\end{matrix}\right\}Cl_3$ it could be proved that also under the prevailing conditions this salt is rather unstable, exactly as was previously mentioned.

From the least-soluble *chloro-d-tartrate* a *chloride* was obtained, which proved to be *ditetragonal*, showing $m = \{110\}$ predominant and $o = \{111\}$; taking $m = \{100\}$ and $o = \{101\}$, these crystals prove to be identical with those of the *racemic* salt: $\{Co(r\text{-}Chxn)_3\}Cl_3 + 1H_2O$, with: $a:c = 1:0{,}7212$; they are optically uniaxial, negative. Evidently the original salt is, owing to the action of the solvent, primarily transformed into a mixture of $\{Co(r\text{-}Chxn)_3\}Cl_3 + 1H_2O$ and $\{Co(Ene)_3\}Cl_3 + 3H_2O$; then the two salts, by the action of the *Ag-d-tartrate*, are partially resolved into the least-soluble *chloro-d-tartrates*, from which, indeed, another chloride could be isolated, the rotation of which was identical with that of $L\text{-}\{Co(d\text{-}Chxn)_3\}Cl_3 + 3H_2O$. There is every reason, therefore, to suppose that the salt $\left\{Co\begin{matrix}(r\text{-}Chxn)_2\\(Ene)\end{matrix}\right\}Cl_3$ is, under all circumstances, really rather unstable when in contact with its solution, and that it is decomposed in the way indicated above.

As to the fission-experiments of the salts of the formula: $\left\{Co\begin{matrix}(r\text{-}Chxn)\\(Ene)_2\end{matrix}\right\}X_3$, the following remarks can be made. The fission was tried in the case of the *bromide* ($+ 1H_2O$) by means of the *bromo-d-tartrates* in the usual way.

---

[1]) F. M. JAEGER, Recueil d. Trav. d. Chim. d. Pays-Bas, 38, 205 (1919). The axial ratio of the *triethylenediamine* salt is $a:b:c = 0{,}87:1:1{,}74$.

But in this case no positive results could be obtained either: the difference in the solubilities of the two highly soluble *chloro-* or *bromo-d-tartrates* evidently is too small to allow a fission. Moreover, a treatment of the salt with the hot mother-liquors, with solutions of the *Ag-tartrate*, etc., very rapidly causes it to be decomposed into the *tricyclohexane-* and *triethylenediamine* salts, so that finally these tentatives were stopped.

§ 7. Finally we tried to ascertain also in the case of the optically active *cyclohexanediamines*, in how far these molecules possibly could be introduced into complex ions of this kind, in which optically active molecules are already present. For this purpose we chose the *praseo*-salt: $\left\{ Co \begin{matrix} (l\text{-}Cptn)_2 \\ Cl_2 \end{matrix} \right\} Cl$ and, by heating it alternately with *d-* or *l-cyclohexanediamine* in the theoretical quantity, we tried to produce the compounds: $\left\{ Co \begin{matrix} (l\text{-}Cptn)_2 \\ (d\text{-}Chxn) \end{matrix} \right\} Cl_3$ and $\left\{ Co \begin{matrix} (l\text{-}Cptn)_2 \\ (l\text{-}Chxn) \end{matrix} \right\} Cl_3$ respectively. A detailed study of the compounds thus obtained has taught us, however, that beyond all doubt the products mentioned are *not* stable, neither in the case where three *levo*gyratory molecules are present in the ion, nor in that of the simultaneous presence of two *levo-* and one *dextro*gyratory molecule. The products isolated always were: $\{Co(l\text{-}Cptn)_3\}Cl_3$, besides $\{Co(d\text{-}Chxn)_3\}Cl_3$ or $\{Co(l\text{-}Chxn)_3\}Cl_3$ respectively. Neither proves it possible to prepare the compound $\left\{ Co \begin{matrix} (l\text{-}Chxn)_2 \\ (d\text{-}Chxn) \end{matrix} \right\} Cl_3$; etc.

The decomposition of the primarily formed salts by the contact with the solute evidently takes place in exactly the same way as was observed by us in all analogous cases, namely according to the reaction-scheme: $3 \left\{ Me \begin{matrix} X_2 \\ Y \end{matrix} \right\} Cl_3 \rightarrow 2\{Me\, X_3\}Cl_3 + \{Me\, Y_3\}Cl_3$. This fact thus once more confirms the strong tendency of such complex ions to assume the possibly highest symmetrical configurations of identical substitutes round their central atom.

Groningen, Laboratory for Inorganic and
Physical Chemistry of the University.

Medicine. — *Untersuchungen an den isolierten Augenmuskeln des Kaninchens während des Lagenystagmus bei akuter Alkoholvergiftung.* Von J. LE HEUX und A. DE KLEYN.

(Communicated at the meeting of March 20, 1937).

Im Jahre 1913 fand ROTHFELD [1]), dass Kaninchen während einer akuten Alkoholvergiftung einen Spontannystagmus zeigen, dessen Richtung wechselt je nach der Lage des Kopfes im Raume: fehlend in der Normalstellung des Kopfes ist er besonders deutlich in den beiden Seitenlagen und hat dabei eine ganz bestimmte Richtung. Sowohl in linker wie in rechter Seitenlage des Kopfes schlägt der Nystagmus des oben liegenden Auges mit der schnellen Komponente ohrenwärts. Vor einigen Jahren wurde in einer Untersuchung mit VERSTEEGH [2]) dieser sogenannte Alkoholnystagmus einer näheren Analyse unterzogen und später in der Festschrift für KUBO[3]) über den Einfluss von Sauerstoffinhalation auf diesen Nystagmus und die induktiven Erscheinungen bei demselben berichtet.

Bei diesen ebenso wie bei den ursprünglichen Untersuchungen von ROTHFELD handelte es sich um Nystagmusbewegungen des ganzen Bulbus. Im folgenden werden die Resultate von einigen Untersuchungen an den isolierten Augenmuskeln von Kaninchen während einer akuten Alkoholvergiftung mitgeteilt, wobei besonders der Ablauf dieses Nystagmus und die induktiven Erscheinungen bei demselben näher untersucht wurden.

Als Versuchstiere wurden, wie schon gesagt, Kaninchen benützt. Der M.rectus externus und M.rectus internus eines Auges wurden in Aethernarkose freipräpariert und der Bulbus entfernt. Nach dem Freipräparieren der Muskeln wurde das Grosshirn entfernt, da sich bei früheren Versuchen herausgestellt hatte, dass auch ohne Grosshirn ein schöner Alkoholnystagmus auszulösen ist und auf diese Weise die Aethernarkose während des eigentlichen Versuchs eingestellt werden kann. Nach einer Stunde wurde dem Tier pro Kilogramm 6 ccm 96 % Alkohol, mit Wasser bis zu 100 ccm verdünnt mittels einer Magensonde per os verabreicht. Darauf wurde das ganze Versuchstier sofort in eine der beiden Seitenlagen gebracht und in dieser Lage gelassen, wobei der Kopf sich in normaler Stellung in Bezug auf den Rumpf befand. Nachdem die Augenmuskeln mit Hebeln verbunden worden waren, wurden die Augenbewegungen nach der Methode von TOPOLANSKI-BARTELS auf ein Kymographion registriert. In den Figuren bedeutet ein Aufwärtsgehen der Kurven Kontraktion des Augenmuskels, während die Zeitmarkierung in Sekunden erfolgte.

---

[1]) ROTHFELD, J., Arbeiten aus dem neurol. Institut der Wiener Universität **22**, 1913, 88.

[2]) KLEYN, A. DE und VERSTEEGH, C., Acta Oto-Laryngologica **14**, 1930, 356.

[3]) KLEYN, A. DE, Festschrift für Prof. Dr. INO KUBO, 1934.

Ablauf des Alkoholnystagmus.

Schon früher [1]) wurde über den Ablauf des Alkoholnystagmus bei intaktem Bulbus näher berichtet. In diesen Versuchen wurde, sobald der Nystagmus aufgetreten war, je nach fünf Minuten während einer Minute die Zahl der Nystagmusschläge gezählt. Dabei wurden in fünfzehn Versuchen die folgenden Mittelwerte gefunden: Anfang des Nystagmus 15 Minuten nach Verabreichung des Alkohols, Maximum der Nystagmusschläge pro Minute nach 35 Minuten und Verschwinden des Nystagmus nach 86 Minuten. Die Zahl der Nystagmusschläge war sehr wechselnd, als Maximumwert wurden ungefähr 200 Schläge pro Minute festgestellt. Diese Resultate sind gewiss nicht in dem Sinne zu deuten, dass bei einem bestimmten Gehalt an Alkohol im Blute der Nystagmus anfängt, um beim Steigen dieses Gehalts stärker zu werden und beim Verschwinden des Alkohols aus dem Blute ebenfalls zu verschwinden.

Bei 5 Kaninchen wurde nach Verabreichung von Alkohol in bestimmten Zeitabständen der Alkoholgehalt des Blutes nach der Methode von WIDMARK [2]) bestimmt. Hierbei stellte sich heraus, dass der Alkoholgehalt unmittelbar nach der Verabreichung schnell zunimmt, nach ungefähr zwei Stunden sein Maximum erreicht und erst nach mehreren Stunden allmählich abnimmt. Wie schon oben gesagt, erreicht der Alkoholnystagmus sein Maximum nach ungefähr einer halben Stunde und verschwindet, wenn man das Tier in derselben Seitenlage liegen lässt, schon nach ungefähr anderthalb Stunden; so dass Alkoholgehalt des Blutes und Alkoholnystagmus gewiss nicht parallel verlaufen.

In den Versuchen an den isolierten Augenmuskeln konnte das folgende festgestellt werden:

Fig. 1. Zu Beginn des Alkoholnystagmus ist derselbe langsam und hat einen vestibulären Charakter mit typischer reciproker Innervation.

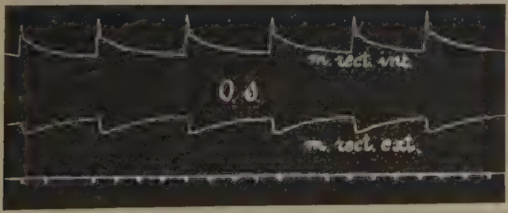


Fig. 1.

Die Tatsache, dass der Alkoholnystagmus wechselt je nach der Lage des Kopfes im Raume, macht es schon wahrscheinlich, dass bei der Genese dieses Nystagmus ein labyrinthärer Faktor eine Rolle spielt. In der früheren Mitteilung mit VERSTEEGH wurden noch verschiedene andere experimentelle Befunde mitgeteilt, welche das Mitwirken eines labyrinthären Faktors bei der Genese dieses Nystagmus höchstwahrscheinlich machen.

---

[1]) Festschrift für KUBO l.c.

[2]) WIDMARK, E., Bioch. Zeitschr. **131**, 1922, 473.

Fig. 2 und 3. Allmählich wird der Nystagmus schneller (Fig. 2) um beim Maximum (Fig. 3) eine sehr grosse Schnelligkeit zu erreichen, wobei auch die Augenmuskelkontraktionen an Grösse zunehmen.

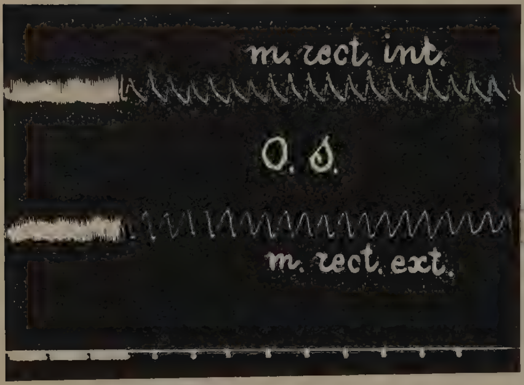


Fig. 2.

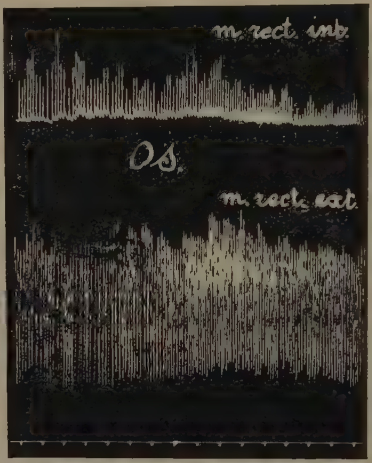


Fig. 3.

Fig. 4. Bevor der Nystagmus, wie schon oben gesagt, nach ungefähr anderthalb Stunden ganz verschwindet, trat in den bis jetzt untersuchten Fällen eine typische Gruppenbildung auf. Nach einer Reihe von Nystagmusschlägen tritt eine ziemlich langdauernde Deviation auf (in diesem Falle Kontraktion des M.rect.ext. und Erschlaffung des M.rect.int.) welche gefolgt wird von einer neuen Reihe von Nystagmusschlägen. Hierauf folgt wieder eine Deviation u.s.w.

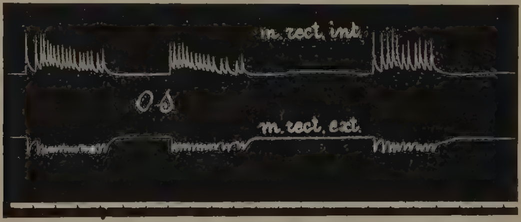


Fig. 4.

Auf die Gruppenbildung beim vestibulären Nystagmus wurde schon früher [1]) hingewiesen. Experimentell wurde dieselbe bis jetzt bei Läsionen des Cerebellum gefunden und auch klinisch kam in einem Fall diese Abweichung zur Beobachtung. Es hat sich später herausgestellt, dass eine derartige Gruppenbildung bei Patienten mit Erkrankungen des Zentralnervensystems häufiger beobachtet werden kann, wenn man nur bei der Untersuchung die Aufmerksamkeit darauf lenkt. Auf die klinische Seite dieser Frage hoffen wir später zurück zu kommen.

---

[1]) KLEYN, A. DE, Archiv. f. Ohrenheilk. 122, 1929, 169.

Die Zeitmarkierung ist in Sekunden, es muss also nicht ausser Acht gelassen werden, dass das Kymographion bei Fig. 4 viel langsamer ging als bei Fig. 1, rechte Hälfte von Fig. 2 und Fig. 3.

Ueber induktive Erscheinungen beim Alkoholnystagmus.

Bei intaktem Bulbus wurde seinerzeit das folgende festgestellt: 1)

Wie schon oben gesagt, zeigen Kaninchen, wenn sie sofort nach der Verabreichung von Alkohol in eine Seitenlage gebracht und in dieser Lage gelassen werden, einen Nystagmus, der nach 15 Minuten anfängt, sein Maximum nach 35 Minuten erreicht und nach 86 Minuten wieder verschwindet. (Durchschnittswerte aus 15 Versuchen).

Ganz anders ist der Verlauf, wenn auf folgende Weise vorgegangen wird: Nach der Verabreichung des Alkohols wird das Versuchstier in eine, z.B. in die linke Seitenlage gebracht und fünf Minuten in dieser Lage gelassen; dann wird es in rechte Seitenlage gelegt und auch in dieser Lage fünf Minuten gelassen. Die Lage wird nach fünf Minuten wieder gewechselt u.s.w. In jeder Seitenlage wird während der fünften Minute die Zahl der Nystagmusschläge genau gezählt.

Als Durchschnittswert wurden auf diese Weise aus vier Versuchen die folgenden Data festgestellt: Anfang des Nystagmus nach 15 Minuten, Maximum erreicht nach 55 Minuten, und Verschwinden des Nystagmus nach 175 Minuten.

Fig. 5 veranschaulicht das Resultat in der Form von einer Kurve. Auf der Abscisse ist die Zeit in Minuten, auf der Ordinate die Zahl der

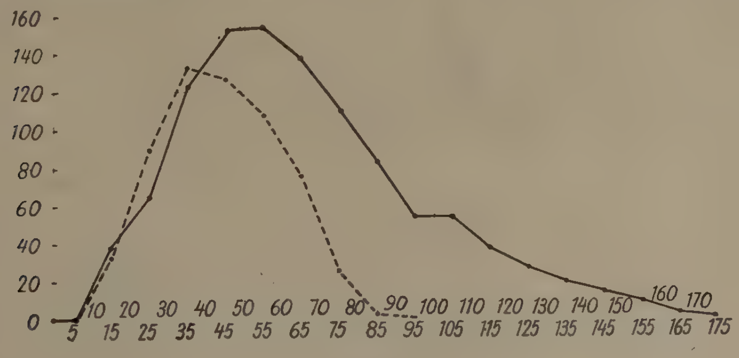


Fig. 5.

---

1) De Kleyn, Festschrift für Kubo. Da diese Festschrift den meisten Lesern nicht zur Verfügung stehen dürfte, möge ihr Inhalt auch hier wiedergegeben werden.

Nystagmusschläge angegeben. Die punktierte Linie stellt die Durchschnittswerte dar von den oben genannten 4 Kaninchen in einer Versuchsreihe, in welcher sie während des ganzen Versuchs in derselben Seitenlage gelassen wurden. Die ausgezogene Linie stellt die Durchschnittswerte dar bei denselben 4 Kaninchen, in einer anderen Versuchsreihe, in welcher die Tiere abwechselnd in linke und rechte Seitenlage versetzt wurden. In die Kurve ist jedoch nur die Zahl der Nystagmusschläge eingezeichnet, welche in linker Seitenlage auftraten.

Hieraus geht hervor, dass der Nystagmus schneller sein Maximum erreicht und eher verschwindet, wenn man die Versuchstiere dauernd in ein und derselben Seitenlage lässt, als wenn man dieselben während des Versuchs abwechselnd in linke und rechte Seitenlage versetzt.

Um eine Erklärung für diese merkwürdige Tatsache zu finden, wurde eine neue Versuchsreihe ausgeführt. Da alle diese Versuche dasselbe Resultat ergaben, möge hier nur ein Versuch als Beispiel angeführt werden.

Fig. 6. Ein Kaninchen wurde nach Verabreichung von 6 ccm 96 % Alkohol pro Kilogramm Tier sofort in die linke Seitenlage gebracht. Nach 10 Minuten stellte sich ein Spontannystagmus ein, der nach 30 Minuten sein Maximum erreichte (100 Schläge pro Minute) und nach einer Stunde verschwand. Als jetzt das Tier in rechte Seitenlage gebracht wurde, trat in dieser Lage noch ein starker Spontannystagmus auf, welcher nach 20 Minuten verschwand. Wurde das Tier nun wieder in linke Seitenlage versetzt, in welcher der Spontannystagmus vorher ganz verschwunden war, so trat derselbe neuerdings auf und hielt 15 Minuten an. Brachte man das Tier nun in rechte Seitenlage, so trat wieder Spontannystagmus auf u.s.w.

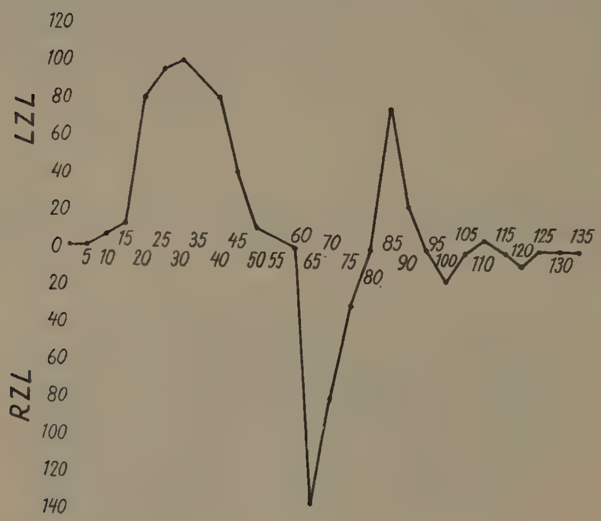


Fig. 6.

Das ganze Phänomen gehört höchstwahrscheinlich zu derselben Gruppe von Erscheinungen, welche wir aus den Untersuchungen von

SHERRINGTON [1]) in Bezug auf die spinalen Reflexe kennen gelernt haben. "If a reflex-arc *A*, during its own activity temporalily checks that of an opposed reflex-arc *B*, but as a subsequent result induces in arc *B* a phase of greater excitability and capacity for discharge, it predisposes the spinal organ for a second reflex opposite in character to its own in immediate succession to itself."

Dass Obiges nicht nur für die spinalen, sondern auch für die vestibulären Reflexe gibt, geht u.E. aus den interessanten Untersuchungen von FISCHER und WODAK [2]) hervor, die bezüglich der Drehreflexe und Drehempfindungen feststellten, dass dieselben einen ganz charakteristischen rhytmischen Ablauf aufweisen. FISCHER und WODAK führen verschiedene Gründe an, die für die zentrale Genese dieses Ablaufs sprechen.

Dass die beim Alkoholnystagmus auftretenden, oben beschriebenen Erscheinungen höchstwahrscheinlich auch zentralen Ursprungs sind, geht aus folgenden Beobachtungen hervor.

Wie schon früher mitgeteilt wurde, zeigen die meisten Kaninchen während einer akuten Alkoholvergiftung wohl einen Spontannystagmus in den beiden Seitenlagen, jedoch nicht bei Normalstellung des Kopfes [3]). Wird nun solch ein Tier in eine Seitenlage gebracht und in dieser gelassen, bis der Spontannystagmus verschwunden ist, so bekommt es nicht nur, wie schon oben mitgeteilt, wieder einen Spontannystagmus, wenn es in die andere Seitenlage gebracht wird, sondern auch schon in Normalstellung. Dieser Befund könnte in Analogie mit der Erklärung SHERRINGTON's in Bezug auf die spinalen Reflexe auf die Weise gedeutet werden, dass durch eine längere Inanspruchnahme des einen Vestibularsystems das antagonistische System in einen Zustand höherer Reizbarkeit gerät. Zum Schluss nimmt die Reizbarkeit dieses Systems dermassen zu, dass es sich schon unter Umständen entladet, unter welchen sonst keine Entladung stattfindet, in unserem Falle also in der Form eines Spontannystagmus in Normalstellung des Tieres.

Bei den Beobachtungen am intakten Bulbus sind die wahrgenommenen Erscheinungen schwer zu analisieren, da bei den Augenbewegungen immer eine Zusammenwirkung von verschiedenen Augenmuskeln stattfindet. Bei Kaninchen haben wir sogar nicht nur mit den sechs gewöhnlichen Augenmuskeln, sondern auch noch mit den vier Retractores bulbi zu rechnen.

Es war darum wünschenswert, zu untersuchen ob es möglich ist, auch am isolierten Augenmuskel diese induktiven Erscheinungen auszulösen. Dass dies gelingt, geht aus den folgenden Versuchen hervor:

---

[1]) SHERRINGTON, CH. S.: The integrative action of the nervous system. London Constable and Co Ltd. S. 212.

[2]) FISCHER, H. M. und WODAK, E., Ausführliche Angaben der Literatur s. KLEYN, A. DE: Statischer Sinn. J. Physiol. 1922. Diese rhytmischen Erscheinungen wurden später durch FISCHER und seine Mitarbeiter in Bezug auf verschiedene labyrinthäre Reflexe bestätigt.

[3]) Sehe KLEYN, A. DE und VERSTEEGH, C., Acta Oto-laryngologica, **14**, 1930, 356.

Fig. 7. Registrierung der Bewegungen des M.rect.ext. des linken Auges. Tier in rechter Seitenlage, Grosshirn exstirpiert. 6 ccm 96 % Alkohol pro Kg. Tier mit Magensonde verabreicht. Im Anfang des Versuchs trat ein deutlicher Alkoholnystagmus der M.rect.ext. auf mit langsamen Kontraktionen und schnellen Erschlaffungen des Muskels. Als der Nystagmus ganz aufgehört hatte, wurde durch galvanische Reizung des linken Labyrinths mit einem Strom von 7 M ampère ein Nystagmus in einer, der ursprünglichen Richtung des Alkoholnystagmus entgegengesetzten Richtung ausgelöst, also mit langsamen Erschlaffungen und schnellen Kontraktionen des M.rect.ext. Nach Aufhören der elektrischen Reizung trat eine kurze Nachwirkung auf, die sich in einem Nystagmus von derselben Richtung wie während der Reizung äusserte, woran sich eine kräftige Kontraktion des M.rect.ext. anschloss, gefolgt von einem längere Zeit dauernden Nystagmus von demselben Typus als der ursprüngliche Alkoholnystagmus.

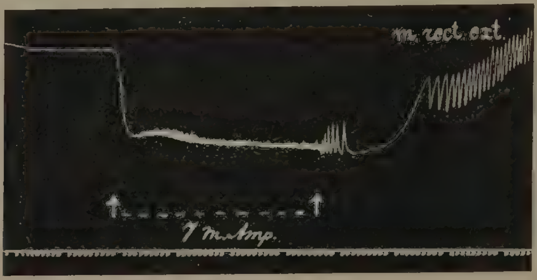


Fig. 7.

Fig. 8 zeigt im Prinzip genau dasselbe. In diesem Versuch wurde, sobald der Alkoholnystagmus beinahe verschwunden und sehr langsam geworden war, durch kalorische Reizung des linken Ohres mit kaltem Wasser ein Nystagmus entgegengesetzter Richtung hervorgerufen. Nachdem die Ausspritzung des linken Gehörganges aufgehört hatte, blieb der kalorische Nystagmus noch kurze Zeit bestehen um dann einem Nystagmus in der ursprünglichen Richtung des Alkoholnystagmus Platz zu machen.

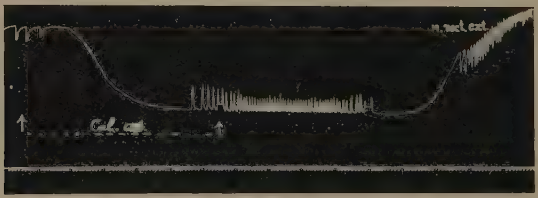


Fig. 8.

Auch bei diesem Versuch haben wir wohl mit den oben beim Alkoholnystagmus besprochenen induktiven Erscheinungen zu tun. Bei oberflächlicher Betrachtung könnte man meinen, dass vielleicht ein anderes Phänomen im Spiele sei[1]. Früher konnte gezeigt werden, dass bei Kaninchen, wenn man die kalorische Reizung eines Labyrinths längere Zeit fortsetzt, und dann mit der Ausspritzung aufhört, der normale Kaltwassernystagmus noch einige Zeit fortdauert, dann aber nicht einfach aufhört, sondern von einem Nystagmus in entgegengesetzter Richtung gefolgt wird. Die Ausspritzung war beim obigen Versuch jedoch von zu kurzer Dauer, um einen derartigen Umschlag des Nystagmus durch kalorische Reizung verursachen zu können. Uebrigens muss u.E. der Umschlag des kalorischen Nystagmus nach langdauernder Reizung ebenfalls durch induktive Erscheinungen der Vestibularapparate erklärt werden.

Zusammenfassung.

1. Während akuter Alkoholvergiftung tritt bei Kaninchen ein Spontannystagmus auf, dessen Richtung wechselt je nach der Lage des Kopfes im Raume (ROTHFELD).

2. Die induktiven Erscheinungen, welche SHERRINGTON für die spinalen Reflexe beschrieben hat und welche den später von FISCHER und WODAK beschriebenen rhytmischen vestibulären Reflexen gleichzustellen sind, können bei Kaninchen während Alkoholvergiftung sehr schön beobachtet werden.

3. Auch an den isolierten Augenmuskeln sind Alkoholnystagmus und die oben genannten induktiven Erscheinungen sehr schön auszulösen. Hierbei wurde kurz vor dem Aufhören des Alkoholnystagmus eine eigenartige Gruppenbildung bei den Nystagmusschlägen beobachtet.

4. Der Alkoholnystagmus eignet sich besonders beim intakten Tier sehr gut für einen Vorlesungsversuch, weil dabei auf einfache Weise ein sogenannter Lagenystagmus hervorgerufen werden kann und ohne schwierige Versuchsanordnung die induktiven Erscheinungen SHERRINGTON's bei Reflexen gezeigt werden können.

---

[1] KLEYN, A. DE und VERSTEEGH, C., Acta Oto-Laryngologica, **6**, 1923, 38.

Physics. — *The* ZEEMAN-*effect of doubly ionized Cerium, Ce III.* By T. L. DE BRUIN, J. N. LIER and H. J. VAN DE VLIET. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of March 20, 1937).

Introduction.

The knowledge of the group of elements of the rare earths is, from spectroscopical point of view, very poor. The older descriptions of these spectra are not very reliable. The formation of the rare earths may be explained as being due to the filling up of the 4 *f*-shell, with three electrons distributed among 5 *d*- and 6 *s*-shells. An application of PAULI's Principle shows that atoms and ions of this type will have for each electron structure a very large number of levels giving rise to innumerable lines. As a matter of fact, the spectra of all rare earths show thousands of lines. In the older descriptions of these spectra the lines due to elements in different stages of ionisation are mixed up. The last years a first attack was made by KING. For several elements of this group *Ce, Pr, Nd,* ... KING made a carefully description of the spectra and investigation of the temperature classification [1]). However, as long as ZEEMAN-effects are lacking for this group of elements, one can not expect that a succesful analysis can be obtained. Important results for several of the rare earths have been obtained by MEGGERS and SRIBNER [2]), MEGGERS and RUSSELL [3]), RUSSELL and KING [4]) and by ALBERTSON [5]). For *Ce I* and *Ce II* data have been published by KARLSON [6]) and HASPAS [7]). LANG [8]) has published an analysis of the *Ce IV* spectrum.

During some time the investigation of the ZEEMAN-effect of the rare earths has been placed on the program of the laboratory "Physica". For some of the elements extensive data have been obtained. This note deals with some of the results for the element Cerium, especially the ZEEMAN-effect of lines belonging to the doubly ionized atom, *Ce III*.

---

[1]) KING: Astr. J. **68**, 194, 1928; **72**, 221, 1930: **74**, 328, 1931; **78**, 9, 1933; **82**, 140, 1935.

[2]) MEGGERS and SRIBNER: Bur. St. J. Res. **5**, 73, 1930. Lu I, II, III.

[3]) MEGGERS and RUSSELL: (unpublished data). Yb I, II.

[4]) RUSSELL and KING: Phys. Rev. **46**, 1023, 1934. Eu I.

[5]) ALBERTSON: Phys. Rev. **45**, 499, 1934; **47**, 370, 1934; Astr. J. **84**, 205, 1936; Eu II, Gd I, Sa I, II.

[6]) KARLSON: Zs. f. Phys. **85**, 482, 1933.

[7]) HASPAS: Zs. f. Phys. **96**, 410, 1935.

[8]) LANG: Can. J. of Res. **13**, 1, 1935; **14**, 127, 1936. Ce IV.

Spectrum of Ce III.

The *Ce III* atom is an example of a two electron spectrum and terms of the configurations s^2, sp, sd, sf, p^2, pd, pf, df, f^2 can be expected. The spectralstructure should be analogous with *Ba I* and *La II* [9]). The main interest of the *Ce III* spectrum is connected with the binding of the different electron configurations.

Arc and spark spectra of two small pieces of cerium metal, used as electrodes, have been obtained and photographed with a HILGER *E I* spectrograph. A group of lines, lying in the ultraviolet between λ 3500 and λ 2900 is greatly enhanced upon passing from the arc to the spark and consists largely of hazy and unsymmetrical lines. KING has made a careful description of the Cerium spectra and ascribes this group of lines to *Ce III*. Using KING's data KALIA [10]) made a partial analysis of the *Ce III* spectrum. From our results it turns out that several of his $\triangle \nu$-differences are real. However the values of quantumnumbers and the interpretations are wrong. The ZEEMAN-effects of this group of strong lines made it possible to identify the terms belonging to the *sf* and *pf*-electron configurations. The energy states which can arise from the *sf*-configuration are the odd terms: $^3F_{234}$ and $^1F_3$. The substitution of p for s-type electrons would give the following terms: $^3G_{345}$ $^3F_{234}$ $^3D_{123}$ $^1G_4$ $^1F_3$ $^1D_2$.

ZEEMAN-*effect of Ce III. Experimental procedure.*

The photographs were made in the third order of a six inch grating with 14437 lines per inch and with a radius of 20 feet in an Eagle mounting. The scale is 0.840 Å per mm.

We used the large WEISS electromagnet of this laboratory. A description can be found in an article by P. ZEEMAN and T. L. DE BRUIN in "Handbuch der Phys. Optik", Bd. II, Gehrcke p. 602. The endplanes of the conical polepieces have a diameter of 1 cm. With a distance of these planes of about 5 mm and a current of 110 Amp. the fieldstrength amounted to 38500 Gauss.

The lightsource was a vacuumtrembler of BACK. We used the Cerium-metal (99 % pure) in the form of small pieces with a diameter of 5 mm and a thickness of 1 mm, riveted to a copper strip, which is screwed on the electrodeholder. This procedure is very economical, because the metal is used up entirely.

The conditions for obtaining the Cerium III spectrum were found to be as follows: Airpressure 1 cm; currentstrength 1 amp. and a tension of about 60 Volts. By altering the conditions we also could obtain a strong Cerium II and Cerium I spectrum, whereas the Cerium III lines disappeared.

The two states of polarisation were separated by means of a calcspar rhomb. The temperature of the grating was kept constant to 0.01° C. by

---

[9]) RUSSELL and MEGGERS: Bur. St. J. Res. **9**, 625, 1932.

[10]) KALIA: Ind. J. of Phys. **17**, 137, 1933.

TABLE 1.

Int.	λ	ν_{vac}	Observed ZEEMAN-effect	Remark
80	3543.999	28208.67	(0) 1.11	1
100	3504.596	28525.80	(0.11) (0.25) 0.40) 0.80 0.91 1.03 1.18 1.29 1.41	2
60	3497.755	28581.65	(0.21) (0.42) (0.63) (0.84) 0.44 0.64 0.86 1.06 1.25 1.47 1.69 1.90	3
300	3470.894	28802.84	(0.15) (0.37) (0.59) 0.46 0.66 0.85 1.05 1.26 1.46	4
200	3459.374	28898.75	(0) 1.52	5
150	3454.368	28940.58	(0) (0.25) (0.50) 0.57 0.82 1.07 1.32 1.57	6
150	3443.609	29031.00	(0) (0.21) (0.42) 0.47 0.66 0.87 1.08 1.30	7
125	3427.332	29168.90	(0.16) (0.30) 0.52 0.67 0.81 0.99	8
20	3398.910	29412.79	(0.24) 1.00	9
50	3395.735	29440.33	(0) 1.19	
150	3353.262	29813.19	(0.55) (0.69) 1.26	10
400	3228.564	30964.65	(0.45^5) 1.20	11
300	3147.05	31766.62	(0.33) (0.52) 1.10	12
200	3143.956	31797.84	(0.32) 1.00	13
250	3141.247	31825.27	(0) 1.10^5	
400	3121.548	32026.11	(0) (0.26) (0.51) — 0.65 0.94 1.20 1.48	
200	3110.516	32139.67	(0) 1.31	
200	3106.974	32176.39	(0) 1.14	14
200	3085.089	32404.58	(0.30) (0.65) 0.98 1.27	
100	3057.575	32696.24	(0) 0.80	15
200	3057.214	32700.09	(0) 1.09	16
125	3056.556	32707.03	(0) 0.99	17
600	3055.585	32717.53	(0) 1.08	18
500	3031.559	32976.75	(0) 1.25	19
200	3022.736	33072.96	(0.73) 1.13	20
50	2973.716	33618.13	(0.51) 1.61	21
100	2931.558	34101.58	(0) 1.01	22
80	2925.283	34174.79	(0) 0.88	23
100	2923.84	34191.62	(0) 1.07	24

REMARKS TO TABLE 1:

1. Sharp triplet.
2. Only measurable on shortly exposed plates. Completely resolved in 18 components.

3. Completely resolved in 24 components.
4. Completely resolved in 18 components.
5. Diffuse triplet. Central π component broad. Decrease intensity to the outside. σ-components broad. Decrease itensity to the inside.
6. Completely resolved in 15 components.
7. Completely resolved in 15 components.
8. Completely resolved in 12 components.
9. 10., 11. 12. 13. σ -components diffuse. Settings on the centroid.
15. 16. 23. 24. Diffuse triplet.
14. 17. Fairly sharp triplet.
18. Strong pseudo triplet.
19. Very diffuse triplet.
20. 21. Quartet. Settings on the centroids.
22. Sharp triplet.

using the well-known principle first described in the article by P. ZEEMAN and T. L. DE BRUIN mentioned above.

The plates used were Ilford special rapids. The time of exposure varied from ½ to 6 hours.

sf-fp multiplet. g-values.

Table 1 contains the observed ZEEMAN-effect of 29 lines of *Ce III*. 25 of these lines can be placed in the $6s\,4f$—$6p\,4f$ multiplet of table 2. The situation of terms in both electronconfigurations indicates that the deviation from the RUSSELL—SAUNDERS coupling is large and anomalous g-values can be expected. A quantum mechanical treatment of the fine structure due to the interaction of one s-electron with one arbitrary electron has been made by HOUSTON [11]). From the terms it is possible to calculate the g-values. Applying HOUSTON's formulas to the sf-configuration in *Ce III* one gets:

$$g\,{}^1F_3 = 1 + \frac{1}{2L(L+1)} - \frac{x+1}{2L(L+1)\left[(x+1)^2 + 4L(L+1)\right]^{1/2}} = 1{,}026$$

$$g\,{}^3F_4 = \frac{L+2}{L+1} = 1{,}250.$$

$$g\,{}^3F_3 = 1 + \frac{1}{2L(L+1)} + \frac{x+1}{2L(L+1)\left[(x+1)^2 + 4L(L+1)\right]^{1/2}} = 1{,}058$$

$$g\,{}^3F_2 = \frac{L-1}{L} = 0{,}667.$$

The observed g-values are: $g\,{}^1F_3 = 1.03^5$; $g\,{}^3F_4 = 1.27$; $g\,{}^3F_3 = 1.07$; $g\,{}^3F_2 = 0.66^5$. These g-values can be determined from completely resolved patterns. This is not the case with all the g-values of the fp-terms. In some cases, starting with the completely resolved patterns and then applying the

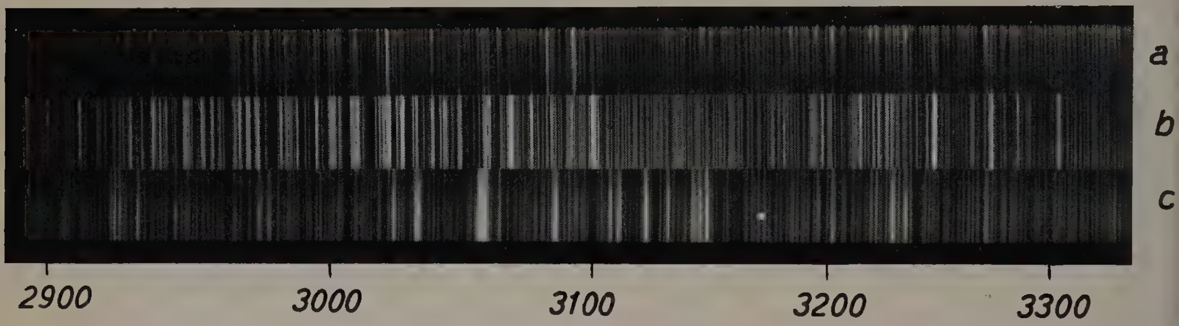
---

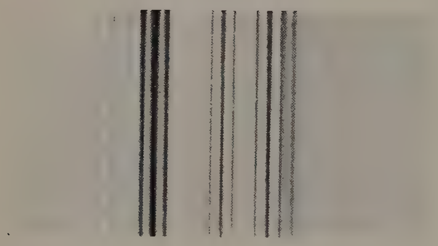
[11]) HOUSTON: Phys. Rev. **33**, 297, 1929.

TABLE 2.

No.	Rel. Term value			0		228.23		2240.10		2613.15
		Term	g	$^3F_2$ 0.66^5		$^3F_3$ 1.07		$^3F_4$ 1.27		$^1F_3$ 1.03^5
1	29031.03	$^3G_3$	0.87	150 3443.609 29031.00	228.16	300 3470.894 28802.84		26790.93		26417.88
2	29168.85	$^3F_2$	0.82	125 3427.332 29168.90	228.32	150 3454.368 28940.58				26555.70
3	30821.79	X_4	1.06			4 3267.72 30593.59	2011.94	60 3497.755 28581.65	372.98	80 3543.999 28208.67
4	31138.90	$^1F_3$	1.16	3 3210.48 31139.03		30910.67		200 3459.374 28898 75	372.95	100 3504.596 28525.80
5	32026.04	$^3F_3$	0.95	400 3121.548 32026.11	228.27	200 3143.956 31797.84		29785.94		20 3398.910 29412.79
6	32053.44	$^3F_4$	1.09			200 3141.247 31825.27	2012.08	150 3353.262 29813.19	372.86	50 3395.735 29440.33
7	32404.60	$^3D_2$	0.99	200 3085.089 32404.58	228.19	200 3106.974 32176.39				29791.45
8	32696.24	$^3D_1$	0.40	100 3057.575 32696.24						
9	33204.87	$^3G_4$	1.13^5			500 3031.559 32976.75	2012.10	400 3228.564 30964.65	372.93	6 3267.92 30591.72
10	34379.81	$^3D_3$	1.24	34379.81		10 2927.258 34151.67	2012.00	200 3110.56 32139.67	373.05	300 3147.05 31766.62
11	34957.63	$^3G_5$	1.21					600 3055.585 32717.53		
12	35313.24	$^1G_4$	1.05			80 2849.372 35085.19	2012.23	200 3022.736 33072.96	372 87	200 3057.214 32700.09
13	35320.18	$^1D_2$	1.08	35320.18		35091.95				125 3056.556 32707.03

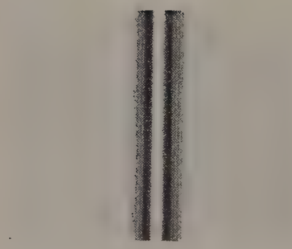
T. L. DE BRUIN, J. N. LIER AND H. J. VAN DE VLIET: THE ZEEMAN-EFFECT OF DOUBLY IONIZED CERIUM, CE III.





1.

2.



3.

4.

Proceedings Royal Acad. Amsterdam, Vol. XL, 1937.

formulas of SHENSTONE and BLAIR [12]) RUSSELL [13]) and CATALAN and POGGIO [14]) to the unresolved blends, the g-values have been derived.

Concerning the sf—pf multiplet a few remarks should be made. In the group of the pf-terms one term with $j = 4$ has been found which finds no place in this group. There is only vacancy for three levels with $j = 4$. Three levels are based on a single combination: $^3D_1$, $^3G_5$ and $^1D_2$. The levels $^3D_1$ and $^3G_5$ seem fairly certain and are based on the ZEEMAN effect. In the place of the line λ 3056.556 it is also possible to put one of the unclassified lines λ 2931.558 or λ 2923.84. The wavenumbers of 9 possible combinations have been calculated, 6 of them are intercombinations.

In conclusion the writers wish to express their appreciation to Professor P. ZEEMAN for his interest and helpful suggestions during this investigation.

Laboratory "Physica" of the University of Amsterdam.

Febr. 1937.

---

[12]) SHENSTONE and BLAIR: Phil. Mag. **8**, 765, 1929.

[13]) RUSSELL: Phys. Rev. **36**, 1590, 1930.

[14]) CATALAN and POGGIO: PIETER ZEEMAN Jubilee Volume. Nijhoff, 1935, p. 387.

DESCRIPTION OF PLATE.

Spectra of cerium:

a. arc spectrum.
b. comparison spectrum of iron.
c spark spectrum.

ZEEMAN-effect, *Ce III.*

1. Pseudotriplets λ 3055,585; 3056,556; 3057,214; 3057,575.
2. λ 3454,368; π- and σ components.
3. λ 3470.894: π- and σ-components.
4. λ 3497,755; π- and σ-components separated.

Physics. — *New terms in the spark spectra of Argon, A. II and A. III.* By T. L. DE BRUIN. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of March 20, 1937).

Singly ionized Argon, A. II.

Some years ago the spectrum of singly ionized Argon has been analyzed by the author ¹). An extensive termtable has been given and it was possible to check several of the terms by ZEEMAN effect ²). Round 600 lines have been classified including practically all the important lines between λ 7500 and λ 2400, and the extreme ultraviolet lines between λ 1974 and λ 487. The early published term table contains the terms arising from the $3p$, $4p$, $3d$, $4d$, $4s$, $5s$, $6s$-electron. A few terms, however, were still lacking namely:

$$3d^4P(^3P),\ 3d^2F_4(^3P),\ 3d^2F(^1D),\ 3d^2G(^1D).$$

An extension of the early analysis seems possible in two directions. In the first place the lacking terms should be detected and in the second place the higher $4f$, $5d$, $5p$-terms are wanted. For the detection of these terms new investigations of the spectrum in the infrared above λ 7500 and in the ultra violet under λ 2400 were necessary. Dr. MEGGERS of the Bureau of Standards made new wavelength measurements in the spark spectrum between λ 7428 and λ 10812, and very courteously placed his data to the disposition of the author. The author has investigated again the ultra violet under λ 2400 with a HILGER *E* 1 quartz spectrograph. Studying the analysis of the spectra of the ionized rare gases the effect of the selfinduction is very useful ³). With the selfinduction in the coil the *A. II* lines are enhanced. Several new *A. II* lines have been detected. With this data an extension of the early analysis is possible. The present paper deals with this extension.

The $3d^4P(^3P)$ term ⁴).

The $3d^4P(^3P)$ levels are:

$$3d^4P_3(^3P) = 76878{,}52$$
$$3d^4P_2(^3P) = 77251{,}38$$
$$3d^4P_1(^3P) = 77526{,}33$$

---

¹) T. L. DE BRUIN, Versl. Kon. Akad. v. Wetensch. Amsterdam, **37**, 340 (1928); Proc. Royal Acad. Amsterdam, **31**, 771 (1928); **33**, 198 (1930); Zs. f. Phys. **48**, 62 (1928); **51**, 108 (1928); **61**, 307 (1930).

²) C. J. BAKKER, T. L. DE BRUIJN, P. ZEEMAN, Proc. Royal Acad. Amsterdam, **37**, 780 (1928); Zs. f. Phys. **52**, 299 (1928); C. J. BAKKER, T. L. DE BRUIJN, Zs. f. Phys. **62**, 32 (1930).

³) T. L. DE BRUIN, PIETER ZEEMAN Verhandelingen, Nijhoff, Den Haag, 413 (1935).

⁴) ROSENTHAL (Ann. d. Physik, **4**, 49 (1930)), made an extensive list of accurate wavelength measurements. He made an extension of the authors analysis by adding the terms: $3d^4P(^3P)$, $5p^4P(^3P)$ and $5p^4D(^3P)$. The terms of ROSENTHAL are however wrong.

Combinations of these new levels with the $4p$-levels are given in table 1. The $3d\,^4P(^3P)$ term is also verified by combinations with new detected $4f$-terms giving lines in the ultraviolet[5]. Several of the infrared lines can be

TABLE 1.

		$3d\,^4P_3$ 76878.52		$3d\,^4P_2$ 77251.38		$3d\,^4P_1$ 77526.33
$4p\,^4S_2\,(^3P)$	63705.60	7 7589.33 13172.77	372.81	4 7380.45 13545.58	275.03	4 7233.58 13820.61
$4p\,^4D_4\,(^3P)$	67520.58	1 10683.20 9357.92				
$4p\,^4D_3\,(^3P)$	67081.22	5 10203.92 9797.48	372.82	4 9829.86 10170.30		
$4p\,^4D_2\,(^3P)$	66586.65	10291.87		10 9374.15 10664.71	275.03	2 9138.48 10939.74
$4p\,^4D_1\,(^3P)$	66326.33			2 9150.77 10925.05	275.02	3 8926.07 11200.07
$4p\,^2D_3\,(^3P)$	66024.28	2 9210.37 10854.35	372.81	2 8904.53 11227.16		
$4p\,^2D_2\,(^3P)$	65361.19	1 8680.11 11517.43		11890.19		3 8217.85 12165.29
$4p\,^2P_2\,(^3P)$	64515.02	2 8086.14 12363.44	372.86	15 7849.42 12736.30	275.04	2 7683.49 13011.34
$4p\,^2P_2\,(^1D)$	52540.70	24337.82		1 4045.71 24710.58	275.14	0 4001.16 24985.72
$4p\,^2D_3\,(^1D)$	51361.07	2 3917.77 25517.52	372.71	1 3861.37 25890.23		

---

[5] For interpretation of the extreme ultra violet lines, See: K. T. COMPTON, J. C. BOYCE and H. N. RUSSELL, Phys. Rev. **32**, 179 (1928); T. L. DE BRUIN, Proc. Royal Acad. Amsterdam, **33**, 208 (1930); J. C. BOYCE, Phys. Rev. **48**, 398 (1935).

TABLE 2.

Int	λ	$\nu_{vac.}$	Termcombination
5	10812.82	9245.75	$3d\ {}^2F_3\ ({}^3P)\ —\ 4p\ {}^2D_2\ ({}^3P)$
1	10683.20	9357.92	$3d\ {}^4P_3\ ({}^3P)\ —\ 4p\ {}^4D_4\ ({}^3P)$
5	10203.92	9797.48	$3d\ {}^4P_3\ ({}^3P)\ —\ 4p\ {}^4D_3\ ({}^3P)$
5	10110.66	9887.84	$4s\ {}^2D_3\ ({}^1D)\ —\ 4p\ {}^2D_3\ ({}^1D)$
100 h	9906.12	10092.00	$3d\ {}^2F_3\ ({}^3P)\ —\ 4p\ {}^2P_2\ ({}^3P)$
1 h	9888.30	10110.19	$4s\ {}^2D_2\ ({}^1D)\ —\ 4p\ {}^2D_3\ ({}^3P)$
4	9829.86	10170.30	$3d\ {}^4P_2\ ({}^3P)\ —\ 4p\ {}^4D_3\ ({}^3P)$
30	9475.20	10550.97	$4s\ {}^2D_3\ ({}^1D)\ —\ 4p\ {}^2D_2\ ({}^3P)$
2	9418.57	10614.41	$3d\ {}^2D_2\ ({}^3P)\ —\ 4p\ {}^2S_1\ ({}^3P)$
10	9374.15	10664.71	$3d\ {}^4P_2\ ({}^3P)\ —\ 4p\ {}^4D_2\ ({}^3P)$
20	9279.72	10773.23	$4s\ {}^2D_2\ ({}^1D)\ —\ 4p\ {}^2D_2\ ({}^3P)$
2	9210.37	10854.35	$3d\ {}^4P_3\ ({}^3P)\ —\ 4p\ {}^2D_3\ ({}^3P)$
2	9150.77	10925.05	$3d\ {}^4P_2\ ({}^3P)\ —\ 4p\ {}^4D_1\ ({}^3P)$
2	9138.45	10939.74	$3d\ {}^4P_1\ ({}^3P)\ —\ 4p\ {}^4D_2\ ({}^3P)$
50	9017.59	11086.40	$4s\ {}^2D_2\ ({}^1D)\ —\ 4p\ {}^2P_1\ ({}^3P)$
3	8926.07	11200.07	$3d\ {}^4P_1\ ({}^3P)\ —\ 4p\ {}^4D_1\ ({}^3P)$
2	8904.53	11227.16	$3d\ {}^4P_2\ ({}^3P)\ —\ 4p\ {}^2D_3\ ({}^3P)$
100	8771.88	11396.94	$4s\ {}^2D_3\ ({}^1D)\ —\ 4p\ {}^2P_2\ ({}^3P)$
1	8680.11	11517.43	$3d\ {}^4P_3\ ({}^3P)\ —\ 4p\ {}^2D_2\ ({}^3P)$
8	8604.04	11619.26	$4s\ {}^2D_2\ ({}^1D)\ —\ 4p\ {}^2P_2\ ({}^3P)$
3	8217.85	12165.29	$3d\ {}^4P_1\ ({}^3P)\ —\ 4p\ {}^2D_2\ ({}^3P)$
2	8086.14	12363.44	$3d\ {}^4P_3\ ({}^3P)\ —\ 4p\ {}^2P_2\ ({}^3P)$
60	8017.55	12469.22	$4s\ {}^2D_2\ ({}^1D)\ —\ 4p\ {}^2S_1\ ({}^3P)$
15	7849.42	12736.30	$3d\ {}^4P_2\ ({}^3P)\ —\ 4p\ {}^2P_2\ ({}^3P)$
2	7683.49	13011.34	$3d\ {}^4P_1\ ({}^3P)\ —\ 4p\ {}^2P_2\ ({}^3P)$
7	7654.06	13061.37	$3d\ {}^2P_2\ ({}^3P)\ —\ 4p\ {}^2D_3\ ({}^3P)$
80	7618.03	13123 15	$4p\ {}^2P_1\ ({}^3P)\ —\ 3d\ {}^2D_2\ ({}^1D)$
250	7589.33	13172.77	$3d\ {}^4P_3\ ({}^3P)\ —\ 4p\ {}^4S_2\ ({}^3P)$
100	7505.13	13320.56	$4p\ {}^2S_1\ ({}^3P)\ —\ 3d\ {}^2P_2\ ({}^1D)$
90	7440.46	13436.33	$4p\ {}^2D_2\ ({}^3P)\ —\ 3d\ {}^2D_2\ ({}^1D)$
1	7428.59	13457.80	$3d\ {}^2P_1\ ({}^3P)\ —\ 4p\ {}^4D_2\ ({}^3P)$
4	7380.45	13545.58	$3d\ {}^4P_2\ ({}^3P)\ —\ 4p\ {}^4S_2\ ({}^3P)$
4	7233.58	13820.61	$3d\ {}^4P_1\ ({}^3P)\ —\ 4p\ {}^4S_2\ ({}^3P)$

already calculated from the early published termtable. Table 2 lists infrared $3d$—$4p$ and $4s$—$4p$ combinations.

The $3d\,{}^4P({}^3P)$ term gives also an explication of the two unclassified *A II* lines in the extreme ultra violet (4) 678,17 and (5) 676,48. It are the combinations of the groundlevel with $3d\,{}^4P_{32}({}^3P)$.

Terms of the 4f-electron.

In *Ne II* 6) and *Kr II* 7) terms have been found which arise from a *f*-electron. Now in *A II* a new set of terms has been found between 29000 and 28000 which combines with the known *3d*-terms. These terms are interpreted as belonging to the *4f*-electrons. The terms are:

$4f\,{}^4F_5({}^3P) = 29954{,}53$
$4f\,{}^4F_4({}^3P) = 29932{,}55$
$4f\,{}^4F_3({}^3P) = 29893{,}33$
$4f\,{}^4F_2({}^3P) = 29757{,}85$
$4f\,{}^4D_4({}^3P) = 29871{,}77$
$4f\,{}^4D_3({}^3P) = 29723{,}47$
$4f\,{}^4D_2({}^3P) = 29456{,}88$
$4f\,{}^4D_1({}^3P) = 29473{,}00$

Combinations of this set of terms with the *3d*-terms are given in table 3. It should be noticed that still more terms of this group have been found namely: 28678,10; 28664,46; 28635,89; 28449,73; 28439,06; 28132,75 and 28121,46. The interpretation of these terms will be reserved for later when we will publish the complete termtable. The *4f*-terms arising from the ${}^1D$ limit can be expected around 16000. One term of this group has been identified: $4f\,{}^2P_2({}^1D) = 16162{,}60$. The term combines with all the known $3d({}^1D)$ terms:

2.	2757,26	36257,19	$3d\ {}^2D_3 — 4f\,{}^2P_2$
1.	2795,45	35761,89	$3d\ {}^2D_2 — 4f\,{}^2P_2$
4.	2924,66	34182,03	$3d\ {}^2P_2 — 4f\,{}^2P_2$
4.	2960,27	33770,87	$3d\ {}^2P_1 — 4f\,{}^2P_2$
0.	3447,40	28999,09	$3da\,{}^2P_2 — 4f\,{}^2P_2$
000.	3488,17	28660,16	$3da\,{}^2P_1 — 4f\,{}^2P_2$
4.	4080,67	24498,89	$3d\ {}^2S_1 — 4f\,{}^2P_2$

It is interesting to note that the Zeeman effect of the line 4080,67 shows an asymetric sextet. The both σ-components to the long wavelength side coincide: 1,10 0,71 **(0,20)** **(0,20)** 0,80.

Doubly ionized Argon, A. III.

The author has given a partial analysis of the *A. III* spectrum 8), including 51 terms belonging to the *3p*, *3d*, *4s*, *4p*, *5s* and *4d* electron, converging to the limits ${}^4S$ and ${}^2D$ of the ion. The two triplet systems are connected by intercombination lines. The quintetsystem was given by

---

6) T. L. de Bruin, Zs. f. Phys. **44**, 157 (1927); **46**, 856 (1928); **69**, 36 (1931).

7) T. L. de Bruin, C. J. Humphreys and W. F. Meggers, Bur. St. J. Res. **11**, 409 (1933).

8) T. L. de Bruin, Proc. Royal Acad. Amsterdam, **36**, 724 (1933); Pieter Zeeman, Verhandelingen, 413 (1935).

TABLE 3.

		$^4F_5$	$^4F_4$	$^4F_3$	$^4F_2$	$^4D_4$	$^4D_3$	$^4D_2$	$^4D_1$
		29954.53	29932.55	29893 33	29757.85	29871.77	29723.47	29456.88	29473.00
3d (^3P) $^4P_3$	76878.52		6 2129.44 46945.82	2 2127.65 46985.31	1 2121.52 47121.05	12 2126 67 47006.96	6 2119.98 47155.28	2 2108.08 47421.43	
3d (^3P) $^4P_2$	77251.38			3 2110.90 47358 09	5 2104.89 47493.29		10 2103.33 47528.52	10 2091.63 47794.34	5 2092.30 47779.03
3d (^3P) $^4P_1$	77526.33				5 2092.78 47768.76			7 2079.69 48069.71	10 2080 36 48053.23
3d (^3P) $^2F_4$	75575.32	3 2191.31 45620.50	45602.77	45681.99		10 2187.35 45703.08			
3d (^3P) $^2F_3$	74606 87		2 2237.71 44674.63	3 2235.75 44713.79	44849.02	15 2234.69 44735.00	8 2227.29 44883.62	45149.99	
3d (^3P) $^2D_3$	73667.15		4 2285.81 43734.64	1 2283.76 43773.90	43909.30	12 2282.63 43795.57	3 2274.93 43943.80	44210.27	
3d (^3P) $^2D_2$	74279.58			6 2252.26 44386.46	44521.73		10 2243.67 44555.97	2 2230.32 44822.64	44806.58
3d (^1D) $^2D_3$	52419.08		1 4445.88 22486.44	1 4438.12 22525.76	22661.23	5 4433.83 22547.56	2 4404.91 22695.59	22962.20	
3d (^1D) $^2D_2$	51924.53			4 4537.67 22031.58	00 4510.00 22166.75		5 4502.95 22201.46	22467.65	22451.53
3d (^1D) $^2P_2$	50344.60			0 4888.29 20451.37	00 4856.20 20586.50		20621.13	2 4786.19 20887.62	20871.60
3d (^1D) $^2P_1$	49933.56				2 4955.18 20175.29			4 4882.29 20476.66	20460.56
3d (^1D) $^2S_1$	40661.40				5 9168 93 10903.41			11204.52	2 8935.44 11188.38
4d (^3P) $^4D_4$	41079.08	150 8986.62 11124.60	25 8968.95 11146.52	2 8937.50 11185.74		3 8920.18 11207.46	2 8803.86 11355.54		
4d (^3P) $^4D_3$	40957.28		100 9068.01 11024.75	20 9035.92 11063.91	2 8926.81 11199.14	11085.51	8 8899.29 11233.77	3 8693.11 11500.21	
4d (^3P) $^4D_2$	40768.67			50 9192.56 10875.38	30 9079.67 11010.60		5 9051.18 11045.25	3 8838.00 11311.67	3 8850.69 11295.46
4d (^3P) $^4D_1$	40562.38				20 9252 63 10804.78			2 9001.66 11106.01	1 9014.97 11089.62
4d (^3P) $^4F_5$	39661.58	10 10299.03 9707.00	1 10275 72 9729.02			9789.81			

estimated values. Boyce 9) has investigated the extreme ultra violet *A. III* spectrum and has related the quintet system to the triplets. Between λ 1973 and λ 395 he could interprete 80 *A. III* lines.

The analysis of the *A. III* spectrum should be extended in two directions. In the first place the third triplet system, converging to the ^2P limit should be developed and in the second place the singlets of the spectrum are lacking. This note concerns the triplet system to the ^2P limit. Our observations of the *A. III* spectrum range from λ 7000 to λ 1900 and are obtained with a 3 m grating (dispersion 5,5 *A p/m . m*) and a Hilger quartz spectrograph E.I. In the earlier paper it has been mentioned how the selfinduction can be used for separation of the lines belonging to the different stages of ionisation.

TABLE 4. Termvalues *A*. III.

Limit ^4S		Limit ^2D		Limit ^2P	
$s^2\ p^4$		$s\,p^5$			
$^3P_2$	329965.80	$^3P_2$	216165.10		
$^3P_1$	328853.40	$^3P_1$	215168.20		
$^3P_0$	328395.60	$^3P_0$	214637 40		
3 d		3 d		3 d	
$^5D_4$	185058.80	$^3G_5$	. . . . .	$^3F_4$	. . . . .
$^5D_3$	185072.85	$^3G_4$	. . . . .	$^3F_3$	. . . . .
$^5D_2$	185079.83	$^3G_3$	. . . . .	$^3F_2$	. . . . .
$^5D_{10}$	185082.87				
		$^3F_4$	143563.65	$^3D_3$	119753.54
$^3D_3$	173048.18	$^3F_3$	143308.60	$^3D_2$	118960.95
$^3D_2$	173041.12	$^3F_2$	143062.75	$^3D_1$	118401.97
$^3D_1$	172934 40				
		$^3D_3$	141251.75	$^3P_2$	116014.93
		$^3D_2$	142142.75	$^3P_1$	115619.10
		$^3D_1$	142794.68	$^3P_0$	115397.31
		$^3P_2$	. . . . .		
		$^3P_1$	141448.48		
		$^3P_0$	. . . . .		
		$^3S_1$	125238.33		

---

9) J. C. Boyce, Phys. Rev. 48, 398 (1935).

TABLE 4. Termvalues *A*. III. (*Continued*).

Limit ^4S		Limit ^2D		Limit ^2P	
4 s		4 s		4 s	
$^5S_2$	155590.80	$^3D_3$	133286.00	$^3P_2$	122732.71
$^3S_1$	149286.80	$^3D_2$	133351.89	$^3P_1$	122433.65
		$^3D_1$	133376.60	$^3P_0$	122292.64
4 p		4 p		4 p	
$^5P_3$	125168.43	$^3F_4$	103319.74	$^3D_3$	89674.14
$^5P_2$	125316.56	$^3F_3$	103462.58	$^3D_2$	89708.21
$^5P_1$	125402.27	$^3F_2$	103609.84	$^3D_1$	89815.14
$^3P_2$	120813.98	$^3D_3$	104563.21	$^3P_2$	86540.83
$^3P_1$	120838.76	$^3D_2$	104817.87	$^3P_1$	86820.04
$^3P_0$	120799.45	$^3D_1$	104810.62	$^3P_0$	87041.84
		$^3P_2$	98624.00	$^3S_1$	90772.32
		$^3P_1$	98338.50		
		$^3P_0$	98211.00		
4 d		4 d		4 d	
$^5D_4$	83919.23	$^3G_5$	62069.98	$^3F_4$	. . . . . .
$^5D_3$	83929.16	$^3G_4$	62132.60	$^3F_3$	48491.98
$^5D_2$	83932.01	$^3G_3$	62183.70	$^3F_2$	48503.83
$^5D_{10}$	83936.04				
		$^3F_4$	62894.58	$^3D_3$	46046.02
$^3D_3$	77676.78	$^3F_3$	63088.30	$^3D_2$	45869.54
$^3D_2$	77712.11	$^3F_2$	63243.00	$^3D_1$	45847.29
$^3D_1$	77692.88	$^3D_3$	60965.00	$^3P_2$	47866.66
		$^3D_2$	60953.00	$^3P_1$	47965.54
		$^3D_1$	60987.00	$^3P_0$	48017.92
		$^3P_2$	58457.92		
		$^3P_1$	58293.72		
		$^3P_0$	58269.58		
		$^3S_1$	57897.35		
5 s		5 s		5 s	
$^5S_2$	79253.53	$^3D_3$	57714.90	$^3P_2$	43956.59
		$^3D_2$	57777.64	$^3P_1$	44083.80
$^3S_1$	77389.92	$^3D_1$	57837.98	$^3P_0$	44134.60

Table 4 lists the terms of *A. III* known up to the present. The new terms to the ^2P limit are related to the other terms by the combinations:

3.	4198,83	23809,47	$4p^3P_2(^2D) — 4s^3P_1(^2P)$
2.	4173,69	23952,98	$4p^3P_1(^2D) — 4s^3P_0(^2P)$
3.	4149,03	24095,25	$4p^3P_1(^2D) — 4s^3P_1(^2P)$
5.	4146,70	24108,78	$4p^3P_2(^2D) — 4s^3P_2(^2P)$
4.	4127,19	24222,73	$4p^3P_0(^2D) — 4s^3P_1(^2P)$
4.	4098,19	24394,15	$4p^3P_1(^2D) — 4s^3P_2(^2P)$

and the combinations with the $4s^3D_{321}\,(^2D)$. The complete list of classified lines

TABLE 5.

		$^3S_1$ 90772.32	$^3D_3$ 89674.14	$^3D_2$ 89708.21	$^3D_1$ 89815.14	$^3P_2$ 86540.83	$^3P_1$ 86820.04	$^3P_0$ 87041.84
4 s $^3D_3$ (^2D)	133286.00		4 2292.25 43611.78	3 2294.05 43577.57		10 2138.59 46744 98		
4 s $^3D_2$ (^2D)	133351.89	42579.57	2 2288.82 43677.13	6 2290.61 43643.00	4 2296.24 43536.01	46811.06	5 2148.38 46532.00	
4 s $^3D_1$ (^2D)	133376.60	42604.28		4 2289.31 43667.78	5 2294.91 43561.24	46735.77	46556.56	3 2157.53 46334.46
3 d $^3S_1$ (^2D)	125238.33	34466.01		35530.12	35423.19	3 2583.39 38697 29	1 2602.12 38418.73	1 2617.26 38196 51
3 d $^3D_3$ (^2P)	119753.54		9 3323.59 30079.32	4 3327.34 30045.42		10 3010.02 33212.72		
3 d $^3D_2$ (^2P)	118960 95	28188.63	6 3413.53 29286.82	7 3417.49 29252.88	2 3430.03 29145.94	3 3083.64 32419.82	7 3110.41 32140.81	
3 d $^3D_1$ (^2P)	118401.97	27629.65		3 3484.12 28693.47	4 3497.10 28586.97	31861.14	31581.93	6 3187.90 31359.75
3 d $^3P_2$ (^2P)	116014.93	8 3960.53 25242.03	20 3795.37 26340.44	6 3800.25 26306.62	1 3815.70 26200 10	15 3391.85 29474.01	9 3424.25 29195.13	
3 d $^3P_1$ (^2P)	115619.10	6 4023.60 24846.37		10 3858.32 25910.70	4 3874.22 25804.36	8 3438.04 29078.03	9 3471.32 28799.27	6 3498.31 28577.09
3 d $^3P_0$ (^2P)	115397.31	3 4059.89 24624.28			7 3907.84 25582.36		6 3498.31 28577.09	
4 s $^3P_2$ (^2P)	122732.71	7 3127 90 31961.10	12 3024.05 33058.64	5 3027.16 33024.67	3 3036.96 32918.11	7 2762.23 36191.96	5 2783.65 35913.48	
4 s $^3P_1$ (^2P)	122433.65	5 3157.42 31662.29		12 3054.82 32725.66	10 3064.77 32619.42	5 2785.23 35893.11	4 2807.02 35614.50	6 2824.66 35392.10
4 s $^3P_0$ (^2P)	122292.64	2 3171.64 31520.34			10 3078 15 32477.64		6 2818.26 35472.47	

will be published later. Table 5 contains important combinations with the set of $4p(^2P)$ levels.

A few remarks should be made to the table 4 concerning to the identification of terms in connection with the series limits. The limits are known from the deep *A. IV* levels [10]).

$$^4S-{}^2D=21167 \qquad {}^2D-{}^2P=13808.$$

The centroids of the groups of multiplets in *A. III* belonging to the different electron configurations should show roughly these distances. Concerning the $^4S-{}^2D$ distance the agreement is fairly well. For the $4p$-, $4s$- and $5s$-electron the distances are: 19133, 20166 and 20875. As can be expected the agreement is better for the higher terms. Concerning the $^2D-{}^2P$ the distances for the $4p$- and $5s$-electron agree giving 15779 and 13758. The $4s^3P(^2P)$, however, is out of position. The $3d$-terms also agree badly with the limits. These points require further investigation.

In conclusion the writer wishes to express his appreciation to Professor P. ZEEMAN for his interest and helpful suggestions during the investigation.

Laboratory "Physica" of the University of Amsterdam.

Febr. 1937.

---

10) J. C. BOYCE, Phys. Rev. **48**, 398 (1935); T. L. DE BRUIN, Physica, III, 8, 809 (1936).

Mathematics. — *Eine arithmetische Eigenschaft gewisser ganzer Funktionen.* III. Von J. POPKEN. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of March 20, 1937).

§ 4. *Beweis des Hilfssatzes C.*

In diesem Paragraphen wird angenommen, dass $x=0$ für die Differentialgleichung $Df(x)=0$ regulär ist, so dass nach der Einleitung die Zahlen $a_\sigma=f^{(\sigma)}(0)$ $(\sigma=0, 1, ..)$ alle von Null verschieden sind und

$$C_6\,\sigma^{n-1} \leqq \left|\frac{a_{\sigma-1}}{a_\sigma}\right| \leqq C_7\,\sigma^{n-1} \qquad (\sigma=1, 2, ..), \;.\;.\;.\;.\; (5)$$

$$\frac{C_6^{\sigma+1}}{\sigma!^{n-1}} \leqq |a_\sigma| \leqq \frac{C_7^{\sigma+1}}{\sigma!^{n-1}} \qquad (\sigma=0, 1, ..) \;.\;.\;.\;.\; (9)$$

gelten. Ferner bezeichnen $C_1, C_2, \ldots, C_{16}$ positive Zahlen, die nur von der Wahl von $f(x)$ abhängen.

Wird

$$-\sum_{\lambda=0}^{n-\nu} \binom{\sigma}{\lambda} P_{\nu+\lambda}^{(\lambda)}(0) = R_\nu(\sigma) \quad (\nu = 1, 2, \ldots, n) \quad . \quad . \quad . \quad (21)$$

gesetzt, so folgt aus Formel (4) der Einleitung mit $x = 0$

$$a_\sigma = \sum_{\nu=1}^{n} R_\nu(\sigma)\, a_{\sigma+\nu} \qquad (\sigma = 0, 1, \ldots) \quad . \quad . \quad . \quad (22)$$

Hierin ist $R_\nu(\sigma)$ offenbar ein Polynom in σ, höchstens vom Grade $n-\nu$; also

$$|R_\nu(\sigma)| \leqq C_8 (\sigma + 1)^{n-\nu} \qquad (\sigma = 0, 1, \ldots) \quad . \quad . \quad . \quad (23)$$

Bedeuten $F(x), G(x)$ zwei Polynome mit $DF(x) = G(x)$, so gilt nach Formel (7) der Einleitung mit $x = 0$

$$F^{(\sigma)}(0) = \sum_{\nu=1}^{n} R_\nu(\sigma)\, F^{(\sigma+\nu)}(0) + G^{(\sigma)}(0) \qquad (\sigma = 0, 1, \ldots).$$

Man nehme insbesondere $G(x) = x^s$, wo $s \geqq 0$ eine ganze Zahl ist, und für $F(x)$ das nach Hilfssatz A eindeutig bestimmte Polynom mit $DF(x) = x^s$. Nach der Bemerkung zu Hilfssatz 2 hat $F(x)$ dann den Grad s; wird also

$$F(x) = \sum_{\sigma=0}^{s} \frac{A_\sigma}{\sigma!} x^\sigma \quad , \quad A_{s+1} = A_{s+2} = \ldots = 0$$

gesetzt, so folgt

$$A_s = s! \quad , \quad A_\sigma = \sum_{\nu=1}^{n} R_\nu(\sigma)\, A_{\sigma+\nu} \qquad (\sigma = 0, 1, \ldots, s-1). \qquad (24)$$

Hilfssatz 5: *Bezeichnen* $R_\nu(\sigma)$ $(\nu = 1, 2, \ldots, n)$ *die Polynome* (21) *und wird für* $\sigma = 0, 1, \ldots$

$$r_n(\sigma) = R_n(\sigma), \; r_\nu(\sigma) = R_\nu(\sigma) + \frac{a_{\sigma+\nu+1}}{a_{\sigma+\nu}} r_{\nu+1}(\sigma) \quad (\nu = n-1, n-2, \ldots, 1) \; (25)$$

gesetzt (man beachte $a_{\sigma+\nu} \neq 0$), *so gilt*

$$r_1(\sigma) = \frac{a_\sigma}{a_{\sigma+1}} \qquad (\sigma = 0, 1, \ldots).$$

Beweis: Für $\nu = 1, 2, \ldots, n-1$ und $\sigma = 0, 1, \ldots$ ist nach (25)

$$r_\nu(\sigma)\,a_{\sigma+\nu} = \left(R_\nu(\sigma) + \frac{a_{\sigma+\nu+1}}{a_{\sigma+\nu}}\, r_{\nu+1}(\sigma)\right) a_{\sigma+\nu} = R_\nu(\sigma)\,a_{\sigma+\nu} + r_{\nu+1}(\sigma)\,a_{\sigma+\nu+1};$$

also

$$\begin{aligned} r_1(\sigma)\,a_{\sigma+1} &= R_1(\sigma)\,a_{\sigma+1} + r_2(\sigma)\,a_{\sigma+2} \\ &= R_1(\sigma)\,a_{\sigma+1} + R_2(\sigma)\,a_{\sigma+2} + \ldots + R_{n-1}(\sigma)\,a_{\sigma+n-1} + r_n(\sigma)\,a_{\sigma+n} \\ &= R_1(\sigma)\,a_{\sigma+1} + R_2(\sigma)\,a_{\sigma+2} + \ldots + R_{n-1}(\sigma)\,a_{\sigma+n-1} + R_n(\sigma)\,a_{\sigma+2} \\ &= a_\sigma, \end{aligned}$$

wegen (22). Die Behauptung ist also richtig.

Hilfssatz 6: *Sei $s \geqq 0$ eine ganze Zahl, $F(x) = \sum_{\sigma=0}^{s} \frac{A_\sigma}{\sigma!} x^\sigma$ das Polynom mit $DF(x) = x^s$, $A_{s+1} = A_{s+2} = \ldots = 0$; wird*

$$\delta_\sigma = A_\sigma - \frac{a_\sigma}{a_{\sigma-1}} A_{\sigma-1} \qquad (\sigma = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

gesetzt, so gilt

$$\sum_{\nu=1}^{n} r_\nu(\sigma)\,\delta_{\sigma+\nu} = 0 \qquad (\sigma = 0, 1, \ldots, s-1),$$

wo die Polynome $r_\nu(\sigma)$ in Hilfssatz 5 definiert sind.

Beweis: Für $\nu = 2, 3, \ldots, n$ und $\sigma = 0, 1, \ldots$ ist

$$\begin{aligned} r_\nu(\sigma)\,\delta_{\sigma+\nu} &= r_\nu(\sigma)\,A_{\sigma+\nu} - \frac{a_{\sigma+\nu}}{a_{\sigma+\nu-1}}\, r_\nu(\sigma)\,A_{\sigma+\nu-1}, && \text{wegen (26),} \\ &= r_\nu(\sigma)\,A_{\sigma+\nu} - \Big(r_{\nu-1}(\sigma) - R_{\nu-1}(\sigma)\Big) A_{\sigma+\nu-1}, && \text{wegen (25),} \\ &= r_\nu(\sigma)\,A_{\sigma+\nu} - r_{\nu-1}(\sigma)\,A_{\sigma+\nu-1} + R_{\nu-1}(\sigma)\,A_{\sigma+\nu-1}; \end{aligned}$$

also

$$\begin{aligned} \sum_{\nu=2}^{n} r_\nu(\sigma)\,\delta_{\sigma+\nu} &= r_n(\sigma)\,A_{\sigma+n} - r_1(\sigma)\,A_{\sigma+1} + \sum_{\nu=2}^{n} R_{\nu-1}(\sigma)\,A_{\sigma+\nu-1} \\ &= R_n(\sigma)\,A_{\sigma+n} + \sum_{\nu=1}^{n-1} R_\nu(\sigma)\,A_{\sigma+\nu} - r_1(\sigma)\,A_{\sigma+1}, \end{aligned}$$

wegen $r_n(\sigma) = R_n(\sigma)$. Ferner ist für $\sigma = 0, 1, \ldots$ nach (26) und Hilfssatz 5

$$r_1(\sigma)\,\delta_{\sigma+1} = r_1(\sigma)\,A_{\sigma+1} - r_1(\sigma)\,\frac{a_{\sigma+1}}{a_\sigma}\,A_\sigma = r_1(\sigma)\,A_{\sigma+1} - A_\sigma.$$

Folglich mit Rücksicht auf (24) für $\sigma = 0, 1, \ldots, s-1$

$$\sum_{\nu=1}^{n} r_\nu(\sigma)\, \delta_{\sigma+\nu} = \sum_{\nu=1}^{n} R_\nu(\sigma)\, A_{\sigma+\nu} - A_\sigma = 0,$$

womit Hilfssatz 6 bewiesen ist.

Hilfssatz 7: *Es gibt ein* C_9, *so dass für die im vorigen Hilfssatz definierten Zahlen* δ_σ *die Ungleichungen*

$$|\delta_\sigma| \leqq C_9^{s+1}\, \sigma! \qquad (\sigma = 1, 2, \ldots)$$

gelten.

Beweis: Nach Hilfssatz 5 ist $r_1(\sigma) = \frac{a_\sigma}{a_{\sigma+1}}$ für $\sigma = 0, 1, \ldots$; also wegen (5)

$$C_6 (\sigma+1)^{n-1} \leqq |r_1(\sigma)| \leqq C_7 (\sigma+1)^{n-1} \quad \ldots \quad (27)$$

Ich werde zeigen, dass bei geeignetem C_{10} sogar

$$|r_\nu(\sigma)| \leqq C_{10} (\sigma+1)^{n-\nu} \qquad (\nu = 1, 2, \ldots, n;\ \sigma = 0, 1, \ldots) \quad . \quad (28)$$

gilt; dazu genügt es für $\nu = 1, 2, \ldots, n$; $\sigma = 0, 1, \ldots$ die Ungleichungen

$$|r_\nu(\sigma)| \leqq C_8 (1 + C_6^{-1} + \ldots + C_6^{-(n-\nu)}) (\sigma+1)^{n-\nu}, \quad \ldots \quad (29)$$

wo C_8, C_6 die Konstanten aus (23), bezw. (5) sind, zu beweisen:

Es ist $r_n(\sigma) = R_n(\sigma)$ (vergleich (25)) und $|R_n(\sigma)| \leqq C_8$ wegen (23), so dass (29) im Fall $\nu = n$ evident ist. Man darf daher voraussetzen, dass $1 \leqq \nu < n$ ist und (29) mit $\nu + 1$ statt ν schon bewiesen ist. Nun ist nach der Definition von $r_\nu(\sigma)$ in (25) für $\sigma = 0, 1, \ldots$

$$|r_\nu(\sigma)| = \left| R_\nu(\sigma) + \frac{a_{\sigma+\nu+1}}{a_{\sigma+\nu}}\, r_{\nu+1}(\sigma) \right| \leqq |R_\nu(\sigma)| + \left| \frac{a_{\sigma+\nu+1}}{a_{\sigma+\nu}} \right| |r_{\nu+1}(\sigma)|.$$

Hierin ist

$|R_\nu(\sigma)| \leqq C_8 (\sigma+1)^{n-\nu}$, wegen (23), $\left| \frac{a_{\sigma+\nu+1}}{a_{\sigma+\nu}} \right| \leqq \frac{1}{C_6 (\sigma+\nu+1)^{n-1}}$, wegen (5),

und schliesslich nach Induktionsvoraussetzung

$$|r_{\nu+1}(\sigma)| \leqq C_8 (1 + C_6^{-1} + \ldots + C_6^{-(n-\nu-1)}) (\sigma+1)^{n-\nu-1};$$

also

$$|r_\nu(\sigma)| \leqq C_8 (\sigma+1)^{n-\nu} + \frac{C_8 (1 + C_6^{-1} + \ldots + C_6^{-(n-\nu-1)}) (\sigma+1)^{n-\nu-1}}{C_6 (\sigma+\nu+1)^{n-1}}$$

$$\leqq C_8 (1 + C_6^{-1} + \ldots + C_6^{-(n-\nu)}) (\sigma+1)^{n-\nu},$$

so dass (29) und damit auch (28) für $\nu = n, n-1, \ldots, 1$ gültig ist.

Nach Hilfssatz 6 ist (man beachte, dass nach Hilfssatz 5 offenbar $r_1(\sigma) = \frac{a_\sigma}{a_{\sigma+1}} \neq 0$ ist):

$$|\delta_{\sigma+1}| = \left| \sum_{\nu=2}^{n} \frac{r_\nu(\sigma)}{r_1(\sigma)} \delta_{\sigma+\nu} \right| \qquad (\sigma = 0, 1, \ldots, s-1),$$

also mit Rücksicht auf (28) und (27)

$$|\delta_{\sigma+1}| \leqq \sum_{\nu=2}^{n} \frac{C_{10}(\sigma+1)^{n-\nu}}{C_6(\sigma+1)^{n-1}} |\delta_{\sigma+\nu}|$$

und daher bei geeignetem C_{11}

$$|\delta_{\sigma+1}| \leqq C_{11} \sum_{\nu=2}^{n} \frac{1}{(\sigma+2)(\sigma+3)\ldots(\sigma+\nu)} |\delta_{\sigma+\nu}| \qquad (\sigma = 0, 1, \ldots, s-1). \quad (30)$$

Ich werde zeigen, dass hieraus

$$|\delta_{\sigma+1}| \leqq C_{12}^{s-\sigma} \frac{(\sigma+1)!}{(s+1)!} |\delta_{s+1}| \qquad (\sigma = 0, 1, \ldots), \quad . \quad . \quad . \quad (31)$$

mit $C_{12} = Max(1, (n-1)\,C_{11})$ folgt. Für $\sigma = s$ ist das evident und ebenfalls für $\sigma = s+1,\ s+2, \ldots,$ wegen

$$\delta_{\sigma+1} = A_{\sigma+1} - \frac{a_{\sigma+1}}{a_\sigma} A_\sigma = 0 - \frac{a_{\sigma+1}}{a_\sigma} 0 = 0.$$

Sei daher $\sigma < s$ und die Ungleichung (31) schon für $\sigma+1,\ \sigma+2, \ldots,$ $\sigma+n-1$ statt σ bewiesen; also mit Rücksicht auf (30)

$$|\delta_{\sigma+1}| \leqq C_{11} \sum_{\nu=1}^{n-1} \frac{|\delta_{\sigma+\nu+1}|}{(\sigma+2)(\sigma+3)\ldots(\sigma+\nu+1)} \leqq C_{11} \sum_{\nu=1}^{n-1} \frac{C_{12}^{s-\sigma-\nu}(\sigma+\nu+1)!\,|\delta_{s+1}|}{(\sigma+2)(\sigma+3)\ldots(\sigma+\nu+1)(s+1)!}$$

$$= C_{11}\, C_{12}^{s-\sigma-1} \frac{(\sigma+1)!}{(s+1)!} |\delta_{s+1}| \sum_{\nu=1}^{n-1} C_{12}^{-(\nu-1)}$$

$$\leqq (n-1)\, C_{11}\, C_{12}^{s-\sigma-1} \frac{(\sigma+1)!}{(s+1)!} |\delta_{s+1}| \leqq C_{12}^{s-\sigma} \frac{(\sigma+1)!}{(s+1)!} |\delta_{s+1}|,$$

wegen $C_{12} \geqq 1$ und $(n-1)\,C_{11} \leqq C_{12}$, womit (31) allgemein bewiesen ist.

Es ist aber wegen $A_{s+1} = 0$, $A_s = s!$ (siehe (24))

$$|\delta_{s+1}| = \left| A_{s+1} - \frac{a_{s+1}}{a_s} A_s \right| = \left| \frac{a_{s+1}}{a_s} \right| s!,$$

so dass nach (5) die Ungleichungen $|\delta_{s+1}| \leqq \frac{1}{C_6 (s+1)^{n-1}} s! \leqq \frac{s!}{C_6}$ gelten und (31) in

$$|\delta_{\sigma+1}| \leqq C_{12}^{s-\sigma} \frac{(\sigma+1)!}{(s+1)!} \frac{s!}{C_6} \qquad (\sigma = 0, 1, \ldots)$$

übergeht, woraus die Behauptung des Hilfssatzes ohne Weiteres folgt.

Hilfssatz 8: *Sei* $s \geqq 0$ *eine ganze Zahl,* $F(x) = \sum_{\sigma=0}^{s} \frac{A_\sigma}{\sigma!} x^\sigma$ *das Polynom mit* $DF(x) = x^s$, *so ist bei geeignet gewähltem* C_{13}

$$|\alpha_0 A_\sigma - \alpha_\sigma A_0| \leqq C_{13}^{s+1} \sigma! \qquad (\sigma = 0, 1, \ldots, s).$$

Beweis: Für $\sigma = 0$ ist die Behauptung klar; sei also $\sigma > 0$, $s > 0$. Nach (6) ist $\frac{1}{|\alpha_\tau|} \leqq \frac{\tau!^{n-1}}{C_6^{\tau+1}}$ für jedes ganze $\tau \geqq 0$, daher folgt aus Hilfssatz 7

$$\left| \frac{A_\tau}{\alpha_\tau} - \frac{A_{\tau-1}}{\alpha_{\tau-1}} \right| = \left| \frac{\delta_\tau}{\alpha_\tau} \right| \leqq C_9^{s+1} \tau! \frac{\tau!^{n-1}}{C_6^{\tau+1}} \leqq C_{14}^{s+1} \tau!^n \qquad (\tau = 1, 2, \ldots, s),$$

falls C_{14} passend gewählt wird; somit gilt für $\sigma = 1, 2, \ldots, s$

$$\left| \frac{A_\sigma}{\alpha_\sigma} - \frac{A_0}{\alpha_0} \right| \leqq \sum_{\tau=1}^{\sigma} \left| \frac{A_\tau}{\alpha_\tau} - \frac{A_{\tau-1}}{\alpha_{\tau-1}} \right| \leqq \sum_{\tau=1}^{\sigma} C_{14}^{s+1} \tau!^n \leqq s\, C_{14}^{s+1} \sigma!^n,$$

also wegen (6)

$$|\alpha_0 A_\sigma - \alpha_\sigma A_0| \leqq |\alpha_0|\, |\alpha_\sigma| \,.\, s\, C_{14}^{s+1} \sigma!^n \leqq |\alpha_0| \frac{C_7^{\sigma+1}}{\sigma!^{n-1}} s\, C_{14}^{s+1} \sigma!^n,$$

so dass die Behauptung folgt.

Hilfssatz 9: *Sei* $s \geqq 0$ *eine ganze Zahl,* $F(x) = \sum_{\sigma=0}^{s} \frac{A_\sigma}{\sigma!} x^\sigma$ *das Polynom mit* $DF(x) = x^s$, *so gilt für jede Zahl* ξ *und jedes* $\nu = 0, 1, \ldots$ *die Ungleichung*

$$|f(0) F^{(\nu)}(\xi) - F(0) f^{(\nu)}(\xi)| \leqq \Gamma_4^{s+1},$$

wo $\Gamma_4 > 0$ *nur von* ξ, ν *und von der Wahl der Funktion* $f(x)$ (also nicht von s) *abhängt.*

Beweis: Es ist

$$f^{(\nu)}(x) = \sum_{\sigma=0}^{\infty} \frac{\alpha_{\sigma+\nu}}{\sigma!} x^\sigma, \quad F^{(\nu)}(x) = \sum_{\sigma=0}^{s-\nu} \frac{A_{\sigma+\nu}}{\sigma!} x^\sigma,$$

also

$$f(0)\,F^{(\nu)}(\xi)-F(0)\,f^{(\nu)}(\xi)=\sum_{\sigma=0}^{s-\nu}\frac{a_0\,A_{\sigma+\nu}-A_0\,a_{\sigma+\nu}}{\sigma!}\,\xi^{\sigma}-A_0\sum_{\sigma=s-\nu+1}^{\infty}\frac{a_{\sigma+\nu}}{\sigma!}\,\xi^{\sigma},\quad(32)$$

wo für $s<\nu$ die Summen $\sum\limits_{\sigma=0}^{s-\nu}$ leer sind, also verschwinden, und $\sum\limits_{\sigma=s-\nu+1}^{\infty}$ gleich $\sum\limits_{\sigma=0}^{\infty}$ gesetzt wird.

Nach Hilfssatz 8 ist

$$\left.\begin{aligned}\left|\sum_{\sigma=0}^{s-\nu}\frac{a_0\,A_{\sigma+\nu}-A_0\,a_{\sigma+\nu}}{\sigma!}\,\xi^{\sigma}\right| &\leqq \sum_{\sigma=0}^{s-\nu}\frac{C_{13}^{s+1}\,(\sigma+\nu)!}{\sigma!}\,|\xi|^{\sigma}\\ &\leqq s\,C_{13}^{s+1}\,s^{\nu}\,(1+|\xi|)^{s}\leqq\Gamma_5^{s+1},\end{aligned}\right\}.\quad(33)$$

für geeignetes nur von ξ, ν und $f(x)$ abhängiges Γ_5.

Ferner folgt aus Hilfssatz 8

$$|a_0\,s!-a_s\,A_0|=|a_0\,A_s-a_s\,A_0|\leqq C_{13}^{s+1}\,s!,$$

also

$$|a_s\,A_0|\leqq C_{13}^{s+1}\,s!+|a_0|\,s!\leqq C_{15}^{s+1}\,s!,$$

für passendes C_{15}. Da ferner nach (6) die Ungleichung $\dfrac{1}{|a_s|}\leqq\dfrac{s!^{n-1}}{C_6^{s+1}}$ gilt, folgt

$$|A_0|\leqq C_{16}^{s+1}\,s!^{n},$$

für geeignetes C_{16}. Daher ist mit Rücksicht auf (6)

$$\left.\begin{aligned}\left|A_0\sum_{\sigma=s-\nu+1}^{\infty}\frac{a_{\sigma+\nu}}{\sigma!}\,\xi^{\sigma}\right| &\leqq C_{16}^{s+1}\,s!^{n}\sum_{\sigma=s-\nu+1}^{\infty}\frac{C_7^{\sigma+\nu+1}}{\sigma!\,(\sigma+\nu)!^{n-1}}\,|\xi|^{\sigma}\\ &\leqq C_{16}^{s+1}\,s!^{n}\,\frac{\Gamma_6^{s+1}}{s!^{n}}=\Gamma_7^{s+1},\end{aligned}\right\}\quad(34)$$

wo Γ_6, Γ_7 positive nur von ξ, ν und $f(x)$ abhängige Zahlen sind.

Aus (32), (33) und (34) folgt

$$|f(0)\,F^{(\nu)}(\xi)-F(0)\,f^{(\nu)}(\xi)|\leqq\Gamma_5^{s+1}+\Gamma_7^{s+1},$$

so dass die Behauptung bewiesen ist.

Hilfssatz C: *Sei $x=0$ für $Df(x)=0$ regulär, sei l eine natürliche Zahl, ferner*

$$G(x)=\sum_{\lambda=0}^{l}g_{\lambda}\,x^{\lambda},\quad G=\sum_{\lambda=0}^{l}|g_{\lambda}|$$

und schliesslich $F(x)$ *das nach Hilfssatz A eindeutig bestimmte Polynom mit* $DF(x) = G(x)$. *Dann gibt es zu jedem* ξ *und jedem* $\nu = 0, 1, \ldots$ *eine nur von* ξ, ν *und von der Wahl der Funktion* $f(x)$ *abhängige Zahl* $\Gamma > 0$, *so dass*

$$| f(0) F^{(\nu)}(\xi) - F(0) f^{(\nu)}(\xi) | \leqq G \Gamma^l$$

ist.

Beweis: Sei $F_\lambda(x)$ das Polynom mit $DF_\lambda(x) = x^\lambda$ $(\lambda = 0, 1, \ldots, l)$, so ist wegen der Additivität des Operators D

$$DF(x) = G(x) = \sum_{\lambda=0}^{l} g_\lambda x^\lambda = \sum_{\lambda=0}^{l} g_\lambda DF_\lambda(x) = D \sum_{\lambda=0}^{l} g_\lambda F_\lambda(x),$$

also

$$F(x) = \sum_{\lambda=0}^{l} g_\lambda F_\lambda(x).$$

Andererseits ist nach Hilfssatz 9

$$| f(0) F_\lambda^{(\nu)}(\xi) - F_\lambda(0) f^{(\nu)}(\xi) | \leqq \Gamma_4^{\lambda+1},$$

wo $\Gamma_4 > 0$ nur von ξ, ν und von der Wahl von $f(x)$ abhängt; also

$$| f(0) F^{(\nu)}(\xi) - F(0) f^{(\nu)}(\xi) | = \Big| \sum_{\lambda=0}^{l} g_\lambda \left(f(0) F_\lambda^{(\nu)}(\xi) - F_\lambda(0) f^{(\nu)}(\xi) \right) \Big|$$

$$\leqq \sum_{\lambda=0}^{l} | g_\lambda | \Gamma_4^{\lambda+1} \leqq G \Gamma^l,$$

bei geeignet gewähltem $\Gamma > 0$, womit die Behauptung bewiesen ist.

§ 5. *Folgerungen des Satzes und Literaturangabe.*

Sei x_0 eine rationale Zahl mit $P_n(x_0) \neq 0$. Aus Formel (4) der Einleitung folgt dann

$$f^{(\sigma)}(x_0) + \sum_{\nu=1}^{n} p_{\sigma\nu} f^{(\sigma+\nu)}(x_0) = 0 \qquad (\sigma = 0, 1, \ldots),$$

mit rationalen $p_{\sigma\nu}$ und insbesondere $p_{\sigma n} = P_n(x_0) \neq 0$. Mit Hilfe dieser Formel lässt sich jeder Ausdruck

$$\sum_{\nu=0}^{N} A_\nu f^{(\nu)}(x_0),$$

mit rationalen A_ν und ganzem $N \geqq 0$ in

$$\sum_{\nu=0}^{n-1} a_\nu f^{(\nu)}(x_0)$$

mit rationalen a_ν überführen. Daher folgt aus dem in dieser Arbeit bewiesenen Satz:

Sind $x_0, x_1, \dots, x_r$ *rationale Zahlen, ist* x_0 *von* $x_1, x_2, \dots, x_r$ *verschieden und regulär für* $Df(x)=0$, $N \geqq 0$ *eine ganze Zahl, sind* $a_{\varrho\nu}$ ($\varrho = 0, 1, \dots, r$; $\nu = 0, 1, \dots, N$) *rationale Zahlen, so folgt aus*

$$\sum_{\varrho=0}^{r} \sum_{\nu=0}^{N} a_{\varrho\nu} f^{(\nu)}(x_\varrho) = 0$$

die Beziehung

$$\sum_{\nu=0}^{N} a_{0\nu} f^{(\nu)}(x_0) = 0.$$

Jedes Polynom $p(x)$ genau vom Grade $n-1$ mit rationalen Koeffizienten lässt sich in der Form

$$p(x) = \sum_{\nu=0}^{n-1} p_{\nu+1}\, x(x-1)\dots(x-\nu+1), \qquad p_n \neq 0,$$

mit offenbar rationalen $p_{\nu+1}$ schreiben. Man zeigt leicht, dass, falls $p(0), p(1), \dots$ alle von Null verschieden sind, die ganze Funktion

$$f_1(x) = 1 + \sum_{\sigma=1}^{\infty} \frac{1}{p(0)\,p(1)\dots p(\sigma-1)} \frac{x^\sigma}{\sigma!}$$

dann der Differentialgleichung

$$f_1(x) - \sum_{\nu=1}^{n} p_\nu x^{\nu-1} f_1^{(\nu)}(x) = 0, \qquad p_n \neq 0, \quad \dots \quad (35)$$

genügt. Beachtet man, dass jede Zahl $x \neq 0$ regulär ist, so folgt durch Anwendung des Satzes: Sind $x_0, x_1, \dots, x_r$ rationale Zahlen, die von einander und von Null verschieden sind, sind $a, a_{\varrho\nu}$ ($\varrho = 0, 1, \dots, r$; $\nu = 0, 1, \dots, n-1$) rationale Zahlen, so folgt aus

$$\sum_{\varrho=0}^{r} \sum_{\nu=0}^{n-1} a_{\varrho\nu} f_1^{(\nu)}(x_\varrho) + a f_1(0) = 0,$$

dass alle Koeffizienten $a_{\varrho\nu}$, also auch a, gleich Null sind. Also:

Ist $p(x)$ *ein Polynom genau vom Grad* n *mit rationalen Koeffizienten, das für* $x = 0, 1, \dots$ *nicht verschwindet, ist*

$$f_1(x) = 1 + \sum_{\sigma=1}^{\infty} \frac{1}{p(0)\,p(1)\dots p(\sigma-1)} \frac{x^\sigma}{\sigma!},$$

und sind $x_0, x_1, \dots, x_r$ *rationale Zahlen, die von einander und von Null* *verschieden sind, so sind die* $(r+1)(n+1)+1$ *Zahlen* $1, f_1^{(\nu)}(x_\varrho)$ $(\varrho=0, 1, \ldots, r;\ \nu=0, 1, \ldots, n)$ *linear unabhängig in bezug auf den Körper der rationalen Zahlen.*

HURWITZ [1]) hat die ganze transzendente Funktion $f_2(x)$, die der Gleichung

$$f_2(x)+a f_2'(x)+b x f_2''(x)=0, \qquad b \neq 0,$$

genügt, und die eng mit BESSELschen Funktionen zusammenhängt, arithmetisch untersucht. Er fand für rationale a, b und rationales $x_0 \neq 0$, dass das Verhältnis $f_2(x_0) : f'_2(x_0)$ irrational ist. Dieses Ergebnis wurde später von STRIDSBERG [2]) und MAIER [3]) verallgemeinert und führte schliesslich zu dem bekannten SIEGELschen Transzendenzsatz über BESSELschen Funktionen [4]).

STRIDSBERG [5]) behandelte ausserdem die ganzen Funktionen $f_1(x)$ mit (35); das STRIDSBERGsche Ergebnis führt also zu der oben genannten Folgerung unseres Satzes. Allgemeiner fand CARLSON [6]): Ist $V(x)$ ein Polynom vom genauen Grad $n+1$ mit rationalen Koeffizienten, das für $x=0, 1, \ldots$ nicht verschwindet, wird

$$\varphi(x)=\sum_{\sigma=0}^{\infty} \frac{x^{\sigma}}{V(0)\,V(1)\ldots V(\sigma)}$$

gesetzt, sind $x_0, x_1, \ldots, x_r$ rationale Zahlen, die von einander und von Null verschieden sind, so sind die $1+(r+1)(n+1)$ Zahlen $1, \varphi^{(\nu)}(x_\varrho)$ $(\varrho=0, 1, \ldots, r;\ \nu=0, 1, \ldots, n)$ linear unabhängig in bezug auf den Körper der rationalen Zahlen. Dieses Resultat ist aber nicht in meinem Satz enthalten.

---

[1]) A. HURWITZ, Ueber arithmetische Eigenschaften gewisser transcendenter Functionen. I, Mathem. Annalen **22** (1883), S. 211—229; II, Mathem. Annalen **32** (1888), S. 583—588. Vergleich auch: I. O. BENDIXON, Sur les équations linéaires à solutions périodiques, Kong. Vetensk. Akad. Oefvers., Stockholm, **53** (1896), S. 193—205, E. RATNER, Ueber eine Eigenschaft gewisser linearer irreductibler Differentialgleichungen, Mathem. Annalen **32** (1888), S. 566—582.

[2]) E. STRIDSBERG, Sur quelques propriétes arithmétiques de certaines fonctions transcendantes, Acta Math. **33** (1910), p. 233—292.

[3]) W. MAIER, Potenzreihen irrationalen Grenzwertes, Journal für d.r.u.a. Math. **156** (1927), S. 93—148, § 6.

[4]) C. L. SIEGEL, Ueber einige Anwendungen diophantischer Approximationen. Abh. Preuss. Akad. d. Wissensch. Jahrgang 1929 (1930), Nr. **1** (70 S.).

[5]) E. STRIDSBERG, Loc. cit., S. 271—282.

[6]) F. CARLSON, Sur une propriéte arithmétique de quelques fonctions entières, Arkiv för mathem., astron. och fysik, **25**A, No. 7 (1935), S. 1—13. Ein zweites Teilergebnis dieses CARLSONschen Satzes findet sich auch schon bei STRIDSBERG (loc. cit., S. 291).

Mathematics. — *On certain infinite sequences.* By CORNELIS VISSER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of March 20, 1937).

§ 1. *Introduction.*

Throughout this paper S denotes a space in which is given a completely additive (non-negative) measure, defined for a complete field of "measurable" sets, to which S itself belongs, and the measure of S is supposed to be 1.

Recently I proved[1])

Theorem 1. *If $E_1, E_2, \ldots$ is an infinite sequence of measurable sets in S, each having a measure not less than m, then for any $\varepsilon > 0$ there exist in this sequence two sets with the property that the measure of their intersection is more than $m^2 - \varepsilon$.*

I shall prove now some further theorems of the same type. From Theorem 1 I shall derive

Theorem 2. *Let $f_1(x), f_2(x), \ldots$ be an infinite sequence of measurable functions on S such that*

$$0 \leqq f_n(x) \leqq 1 \text{ for any } x \text{ in } S \text{ and } n = 1, 2, \ldots.$$

Let a be an accumulation point of the numbers $\int_S f_n(x)\,dx \; (n = 1, 2, \ldots)$

Then, for any $\varepsilon > 0$, there exist two indices p and q $(p \neq q)$ with

$$\int_S |f_p(x) - f_q(x)|\,dx < 2a(1-a) + \varepsilon.$$

I observe that the expression on the right is at most $\frac{1}{2} + \varepsilon$. The exactness of the result is illustrated by the following example. Let $f_n(x)$ $(n = 1, 2, \ldots)$ be defined by

$$f_n(x) = 1 \text{ for } \frac{k}{2^n} \leqq x < \frac{k+1}{2^n} \; (k = 0, 2, \ldots, 2^{n-1}),$$

$$f_n(x) = 0 \text{ elsewhere on the interval } 0 \leqq x \leqq 1.$$

Then $\int_0^1 f_n(x)\,dx = \frac{1}{2}$ for every n and $\int_0^1 |f_p(x) - f_q(x)|\,dx = \frac{1}{2}$ for every p and q $(p \neq q)$.

---

[1]) On POINCARÉ's recurrence theorem, Bulletin of the Am. Math. Soc., 397—400 (1936). After I finished the present paper my attention was drawn to the fact that this theorem was proved also lately by J. GILLIS: Journal of the London Math. Soc. **11**, 139—141 (1936).

The chief object of this paper, however, is the proof of the following more general theorem.

Theorem 3. *Let* $f_1(x), f_2(x), \dots$ *be an infinite sequence of measurable functions on* S *with the properties*

$$0 \leqq f_n(x) \leqq 1 \quad \textit{for any } x \textit{ in } S \textit{ and } n = 1, 2, \dots,$$

$$\int_S f_n(x)\, dx \geqq a \quad \textit{for } n = 1, 2, \dots$$

Then, given $\varepsilon > 0$ *and a natural number* m, *there exists an infinite subsequence*

$$f_{n_1}(x) \;,\; f_{n_2}(x), \dots$$

with the property that for any $r = 1, 2, \dots, m$

$$\int_S f_{n_{i_1}}(x) \dots f_{n_{i_2}}(x)\, dx > a^r - \varepsilon$$

for arbitrary indices $i_1, \dots, i_r$ *no two of which are equal.*

It is easily seen that Theorem 1 is a special case of Theorem 3 with $m = 2$ (every $f_n(x)$ is the characteristic function of a certain set E_n). Also Theorem 2 includes Theorem 1.

I list here also a theorem of a more general nature, which plays an essential rôle in the proof of Theorem 3.

Theorem 4. *Let be given an infinite sequence*

$$a_1 \;,\; a_2 \;,\; a_3 , \dots$$

of arbitrary elements. Suppose that in every infinite subsequence there exists at least one system of n *different elements that has a certain property* P. *Then there exists an infinite subsequence in which every system of* n *different elements has the property* P.

In the paper I referred to above I used Theorem 1 for a simple proof of KHINTCHINE's extension of POINCARÉ's recurrence theorem[1]). By the same method recurrence theorems can be obtained from Theorems 2 and 3. This is carried out in § 5.

§ 2. *Proof of Theorem* 2.

The theorem is trivial if $a = 0$. We may, therefore, suppose that $a > 0$ and also that $\varepsilon < 7a$. The given sequence has an infinite subsequence

$$g_1(x) \;,\; g_2(x), \dots$$

such that, if $y_n = \int_S g_n(x)\, dx$,

$$a - \frac{\varepsilon}{7} < y_n < a + \frac{\varepsilon}{7}.$$

---

[1]) A. KHINTCHINE, Ueber den POINCARÉschen Wiederkehrsatz, Compositio Mathematica 1, 177—179 (1934).

Let S be the "product" of S and the interval $0 \leqq z \leqq 1$. By this is meant the set of all pairs (x, z), where x is a point in S and z a number with $0 \leqq z \leqq 1$. It is well-known that the measure in S and the LEBESGUE measure in the interval $0 \leqq z \leqq 1$ determine a measure in S, the "product measure", in such a way that for any measurable function $\varphi(x)$ on S with $0 \leqq \varphi(x) \leqq 1$ the set

$$0 \leqq z \leqq \varphi(x) \text{ , } x \text{ in } S$$

is measurable and has a measure equal to the value of the integral $\int_S \varphi(x)\, dx$.

Let us consider now the sets defined by

$$0 \leqq z \leqq g_n(x) \text{ , } x \text{ in } S.$$

Each of these sets has a measure $y_n > a - \frac{\varepsilon}{7}$. Then, according to Theorem 1, some two of these sets have an intersection with measure greater than $\left(a - \frac{\varepsilon}{7}\right)^2 - \frac{\varepsilon}{7}$. It is plain that this means that there are two functions $g_p(x)$ and $g_q(x)$ $(p \neq q)$ such that

$$\int_S \min\left(g_p(x), g_q(x)\right) dx > \left(a - \frac{\varepsilon}{7}\right)^2 - \frac{\varepsilon}{7}.$$

Let E be the set on which $g_p(x) \geqq g_q(x)$ and F the set on which $g_p(x) < g_q(x)$. Then we have

$$\int_S |g_p - g_q|\, dx = \int_E (g_p - g_q)\, dx + \int_F (g_q - g_p)\, dx$$

$$= \int_E (g_p - \min(g_p, g_q))\, dx + \int_F (g_q - \min(g_p, g_q))\, dx$$

$$+ \int_E (g_q - \min(g_p, g_q))\, dx + \int_F (g_p - \min(g_p, g_q))\, dx$$

$$= \int_S g_p\, dx + \int_S g_q\, dx - 2 \int_S \min(g_p, g_q)\, dx$$

$$< \left(a + \frac{\varepsilon}{7}\right) + \left(a + \frac{\varepsilon}{7}\right) - 2\left(a - \frac{\varepsilon}{7}\right)^2 + \frac{\varepsilon}{7} < 2a(1-a) + \varepsilon,$$

which proves the theorem.

Remark. The following example shows the precision of our result more fully than the one given in § 1. Let $0 < a < 1$. We define the intervals

$$\begin{array}{c} I_0 \;,\; I_1 \\ I_{00} \;,\; I_{01} \;,\; I_{10} \;,\; I_{11} \\ I_{000} \;,\; I_{001} \;,\; I_{010} \;,\; I_{011} \;,\; I_{100} \;,\; I_{101} \;,\; I_{110} \;,\; I_{111} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{array}$$

as follows: I_0 is the interval $0 \leqq x < a$, I_1 the interval $a \leqq x \leqq 1$. $I_{\alpha_1 \dots \alpha_r 0}$ has the same left end point as $I_{\alpha_1 \dots \alpha_r}$ and its length is a times the length of $I_{\alpha_1 \dots \alpha_r}$, $I_{\alpha_1 \dots \alpha_r 1} = \mathrm{I}_{\alpha_1 \dots \alpha_r} - I_{\alpha_1 \dots \alpha_r 0}$.

We define $f_n(x)$ by:

$$f_n(x) = 1 \qquad \text{if } x \text{ belongs to an interval } I_{\alpha_1 \dots \alpha_n}$$

with $\alpha_n = 0$, and

$$f_n(x) = 0 \qquad \text{if } x \text{ belongs to an interval } I_{\alpha_1 \dots \alpha_n}$$

with $\alpha_n = 1$.

Then, for every $n = 1, 2, \dots$,

$$\int_0^1 f_n(x)\, dx = a,$$

and, for $p \neq q$,

$$\int_0^1 | f_p(x) - f_q(x) |\, dx = 2a(1-a).$$

§ 3. *Proof of Theorem* 4.

The theorem is obviously true if $n = 1$ for any sequence and any property P. For $n > 1$ we shall prove it by means of induction. So assume that the statement is correct if n is replaced by $n-1$, whatever the sequence and the property P may be.

We prove first that then there exists an infinite subsequence

$$a_1', \; a_2', \; a_3', \dots$$

such that every system of n different elements that contains a_1' has the property P.

Let us suppose that this were not true. Then every infinite subsequence of the sequence

$$a_2, \; a_3, \; a_4, \dots \qquad \dots\dots\dots \qquad (1)$$

contains at least one system of $n-1$ different elements with the property that combined with a_1 it gives a system of n different elements that does not have the property P. Hence, by the hypothesis of the induction, the sequence (1) has a subsequence

$$b_2 \ , \ b_3 \ , \ b_4 , \ldots$$

such that any system of $n-1$ different elements of this subsequence has the property that combined with a_1 it gives a system of n different elements that does not have the property P. Now consider the sequence

$$b_3 \ , \ b_4 \ , \ b_5 , \ldots \qquad \ldots\ldots\ldots \qquad (2)$$

By the same argument we find a subsequence

$$c_3 \ , \ c_4 \ , \ c_5 , \ldots \qquad \ldots\ldots\ldots \qquad (3)$$

of (2) such that every system of $n-1$ different elements of (3) has the property that combined with b_2 it gives a system of n different elements not having the property P.

The continuation of this process determines an infinite sequence

$$a_1 \ , \ b_2 \ , \ c_3 , \ldots$$

which obviously has the property that no system of n different elements has the property P. This contradicts the hypothesis of the theorem.

Therefore we find the existence of an infinite subsequence

$$a_1' \ , \ a_2' \ , \ a_3' , \ldots$$

such that any system $a_1', a_{i_1}', \ldots, a_{i_{n-1}}'$ $(1 < i_1 < \ldots < i_{n-1})$ has the property P.

Now consider the sequence

$$a_2' \ , \ a_3' \ , \ a_4' , \ldots$$

By exactly the same argument it follows that this sequence has an infinite subsequence

$$a_3'' \ , \ a_4'' \ , \ a_5'' , \ldots$$

such that any system $a_2'', a_{i_1}'', \ldots, a_{i_{n-1}}''$ $(2 < i_1 < \ldots < i_{n-1})$ has the property P. Then we take the sequence

$$a_2'' \ , \ a_3'' \ , \ a_4'' , \ldots$$

and find an infinite subsequence

$$a_3''' \ , \ a_4''' \ , \ a_5''' , \ldots$$

such that every system $a_3''', a_{i_1}''', \ldots, a_{i_{n-1}}'''$ $(3 < i_1 < \ldots < i_{n-1})$ has the property P. And so on. The infinite sequence

$$a_1' \ , \ a_2'' \ , \ a_3''' , \ldots$$

clearly has the property that any system of n different elements has the property P. Thus the theorem has been proved.

§ 4. *Proof of Theorem 3.*

1. We start with an elementary inequality.

Lemma. Let $0 \leqq a_i \leqq 1$ for $i = 1, \dots, n$. Then, for every $m = 0 \dots, n$,

$$\Sigma\, a_{i_1} \dots a_{i_m} (1 - a_{i_{m+1}} - \dots - a_{i_n}) \leqq 1.$$

The summation has to be taken over all systems of indices $i_1 < \dots < i_m$; in each term the n indices are mutually different.

Proof. We have

$$\Sigma\, a_{i_1} \dots a_{i_m} (1 - a_{i_{m+1}} - \dots - a_{i_n}) \leqq \Sigma\, a_{i_1} \dots a_{i_m} (1 - a_{i_{m+1}}) \dots (1 - a_{i_n})$$

$$\leqq \sum_{m=0}^{n} \Sigma\, a_{i_1} \dots a_{i_m} (1 - a_{i_{m+1}}) \dots (1 - a_{i_n}) = \prod_{k=1}^{n} (a_k + (1 - a_k)) = 1.$$

2. The theorem is true for $m = 1$. To prove it for a number $m > 1$ we shall use the method of induction. So assume that the theorem is true if m is replaced by $m-1$. Let

$$g_1(x)\ ,\ g_2(x), \dots$$

be an infinite subsequence such that

$$\int_S g_{i_1}(x) \dots g_{i_k}(x)\, dx > \tfrac{1}{2} a^k$$

for every system of indices $i_1 < \dots < i_k$ $(k = 1, \dots, m-1)$. We assume here and in the sequal that a is positive; the case $a = 0$ is trivial.

The assumption that for every system of indices $i_1 < \dots < i_m$

$$\int_S g_{i_1}(x) \dots g_{i_m}(x)\, dx \leqq \frac{1}{16\,(m-1)}\, a^m$$

leads to a contradiction. Indeed, substituting $g_i(x)$ for a_i in the inequality of the lemma and taking $r = m - 1$ one would obtain after integration, for $n = m,\ m+1, \dots,$

$$\binom{n}{m-1} \tfrac{1}{2}\, a^{m-1} - (n - m + 1) \binom{n}{m-1} \frac{1}{16\,(m-1)}\, a^m \leqq 1$$

and hence a fortiori

$$(n - m + 2)^{m-1}\, a^{m-1} \left[\tfrac{1}{2} - (n - m + 1) \frac{a}{16\,(m-1)} \right] < (m-1)!$$

The value of n for which

$$\frac{4\,(m-1)}{a} < n - m + 2 \leqq \frac{4\,(m-1)}{a} + 1$$

is greater than m. Substituting this value we find

$$(4\,(m-1))^{m-1}\,[\tfrac{1}{2}-\tfrac{1}{4}] < (m-1)!$$

or

$$(4\,(m-1))^{m-1} < 4\,(m-1)!,$$

which is impossible.

Hence we can conclude that there exists at least one system of indices $i_1 < \ldots < i_m$ such that

$$\int\limits_S g_{i_1}(x)\ldots g_{i_m}(x)\,dx > \frac{1}{16\,(m-1)}\,a^m.$$

3. Now write $\lambda = 1 - \frac{\varepsilon}{a^m}$ and choose a natural number N such that

$$\left(\frac{1}{16\,(m-1)}\right)^{\frac{1}{N}} \geqq \lambda.$$

Let us denote by $\bar{S}$ the so-called N-th power of the space S. This is the space formed by all ordered systems $(x_1, \ldots, x_N)$ of points in S. It is known that the given measure in S determines a measure, the "product measure", in $\bar{S}$ in such a way FUBINI's theorem is valid.

We define in $\bar{S}$ the sequence of functions $F_1, F_2, \ldots$ by

$$F_n(x_1, \ldots, x_N) = f_n(x_1)\ldots f_n(x_N).$$

Then

$$0 \leqq F_n(x_1, \ldots, x_N) \leqq 1,$$

and by FUBINI's theorem

$$\int\limits_{\bar{S}} F_n\,d\bar{x} = \left(\int\limits_S f_n\,dx\right)^N \geqq a^N.$$

Here $d\bar{x}$ denotes the element of integration in $\bar{S}$.

According to the hypothesis of the induction and to what has just been proved, applied to the sequence

$$F_1\,,\ F_2, \ldots,$$

there exists a subsequence

$$G_1\,,\ G_2, \ldots$$

such that

$$\int_{\bar{S}} G_{i_1} \dots G_{i_r}\, d\bar{x} > \tfrac{1}{2}\,(a^N)^r$$

for every $r = 1, \dots, m-1$ and every system of indices $i_1 < \dots < i_r$, and at least one system of indices $i_1 < \dots < i_m$ such that

$$\int_{\bar{S}} G_{i_1} \dots G_{i_m}\, d\bar{x} > \frac{1}{16\,(m-1)}\,(a^N)^m.$$

This means in terms of the space S and the sequence $f_1, f_2, \dots$ that there exists a subsequence

$$g_1\,,\, g_2, \dots$$

such that

$$\left(\int_S g_{i_1} \dots g_{i_r}\, dx\right)^N > \tfrac{1}{2}\,(a^N)^r$$

for every $r = 1, \dots, m-1$ and every system $i_1 < \dots < i_r$, and at least one system $i_1 < \dots < i_m$ such that

$$\left(\int_S g_{i_1} \dots g_{i_m}\, dx\right)^N > \frac{1}{16\,(m-1)}\,(a^N)^m.$$

Therefore

$$\int_S g_{i_1} \dots g_{i_r}\, dx > (\tfrac{1}{2})^{\frac{1}{N}}\, a^r > \lambda\, a^r > a^r - \varepsilon$$

for every $r = 1, \dots, m-1$ and every system $i_1 < \dots < i_r$, and

$$\int_S g_{i_1} \dots g_{i_m}\, dx > \left(\frac{1}{16\,(m-1)}\right)^{\frac{1}{N}} a^m \geqq \lambda\, a^m = a^m - \varepsilon$$

for at least one system $i_1 < \dots < i_m$.

4. The proof is completed by application of Theorem 4. Let us say that the system $g_{i_1}, \dots, g_{i_m}$ of m elements of the sequence

$$g_1\,,\, g_2, \dots$$

has the property P if

$$\int_S g_{i_1} \dots g_{i_m}\, dx > a^m - \varepsilon.$$

We proved that such a system exists. But it is clear that also in every infinite subsequence of $g_1, g_2, \ldots$ at least one such system can be found. Therefore the hypothesis of Theorem 4 is satisfied and we may conclude that $g_1, g_2, \ldots$ has an infinite subsequence in which any system of m elements has the property P. Clearly this subsequence, which is a fortiori a subsequence of the sequence $f_1, f_2, \ldots$, has the properties stated in the theorem.

§ 5. *Application.*

Suppose that we are given a steady incompressible flow in the space S. This means that we have a group of one to one transformations $T_t (-\infty < t < \infty)$ of S into itself with the properties:

(*a*) $T_t T_s = T_{t+s}$,

(*b*) if E is a measurable set then $T_t E$ is measurable and E and $T_t E$ have the same measure.

For the sake of brevity we shall write x_t instead of $T_t x$. If $\varphi(x)$ is any summable function on S, then, by virtue of (*b*),

$$\int_S \varphi(x_t)\, dx = \int_S \varphi(x)\, dx.$$

Let us now consider a measurable function $f(x)$ such that $0 \leqq f(x) \leqq 1$ for every point x and

$$\int_S f(x)\, dx = a.$$

Elementary consequences of Theorems 2 and 3 are the following "recurrence theorems".

For any $\varepsilon > 0$ the inequality

$$\int_S |f(x_t) - f(x)|\, dx < 2a(1-a) + \varepsilon$$

holds for a set of values t which is relatively dense on the t-axis.

For any $\varepsilon > 0$ the inequality

$$\int_S f(x) f(x_t)\, dx > a^2 - \varepsilon$$

holds for a set of values t which is relatively dense on the t-axis.

I shall give only the proof of the first of these theorems, as the proof of the second is similar.

Suppose the theorem were not true. Then we should have

$$\int_S |f(x_t)-f(x)| \geqq 2a(1-a)+\varepsilon$$

on arbitrarily large t-intervals. By induction we could choose a sequence

$$I_1, I_2, \ldots$$

of those intervals in such a manner that the length of I_n were more than

$$2 \max_{1 \leqq k \leqq n-1} \max_{t \text{ in } I_k} |t|.$$

Let c_n be the mid-point of I_n. Then $c_n - c_k$ would be a point of I_n for $k = 1, \ldots, n-1$. Hence

$$\int_S |f(x_{c_n - c_k})-f(x)| \, dx \geqq 2a(1-a)+\varepsilon.$$

It follows, as we observed above, from the incompressibility of the flow that

$$\int_S |f(x_{c_n - c_k})-f(x)| \, dx = \int_S |f(x_{c_n})-f(x_{c_k})| \, dx.$$

Therefore we should have

$$\int_S |f(x_{c_n})-f(x_{c_k})| \, dx \geqq 2a(1-a)+\varepsilon$$

for all natural numbers n and k. This contradicts Theorem 2, applied to the sequence $f(x_{c_1}), f(x_{c_2}), \ldots$ This establishes the theorem.

It is perhaps interesting that our theorems express the recurrence properties of the flow in a non-local manner.

Medicine. — *Further investigations on "healthy" human carriers of Plasmodium vivax in North-Holland.* By N. H. SWELLENGREBEL, A. DE BUCK and H. KRAAN. (From SWELLENGREBEL's Laboratory in the Department of Tropical Hygiene of the Royal Colonial Institute at Amsterdam.) (Communicated by Prof. W. A. P. SCHÜFFNER.)

(Communicated at the meeting of March 20, 1937).

In our paper on the transmission of malaria in North-Holland [1]) we showed that "healthy" parasite carriers are a more important source of anopheline infection than persons actually suffering from malaria and seeking medical assistance as a consequence. The term "healthy" serves no other purpose than to make it clear that these persons are carrying tertian parasites without consulting the physician or taking medicine.

In the present paper we shall try to answer the following questions:

1. Are the parasites in the carriers' blood sufficiently numerous to infect anopheles? The field test yields an answer to the affirmative, but can this be corroborated by the experiment?

2. When writing on healthy carriers we had mainly children in view (persons under 16). What about adults, are they of equal importance in this respect?

1. *Experimental evidence that carriers with few parasites in their peripheral circulation do infect anopheles.*

Anopheles (*maculipennis atroparvus*, North-Holland strain) were allowed once to take their fill on G. P. I.-patients infected with the Madagascar strain of *Plasmodium vivax*, at a time they had entered upon the afebrile period following the regulation series of paroxysms.

The results can be tabulated as follows (Table 1). With their help we can now assess the importance, as a source of anopheline infection, of parasite carriers found in nature.

In 1936 we found in Uitgeest and Wormerveer 103 "healthy" human carriers with the following parasite count:

I:	1	with	50	parasites	per	100	leucocytes,	1	with	male	gametocytes.
II:	7	„	12—25	„	„	„	„	, 3	„	„	„ .
III:	4	„	6— 9	„	„	„	„	, 1	„	„	„ .
IV:	10	„	1— 5	„	„	„	„	, 3	„	„	„ .
V:	22	„	1— 5	„	„	1000	„	, 5	„	„	„ .
VI:	59	„	1— 5	„	„	6000	„	, 1	„	„	„ .

---

[1]) Bulletin trimestriel de l'organisation d'hygiène de la Soc. d. Nations, Vol. V, no. 2, June 1936, pp. 360—369.

As far as the number of parasites goes, the "infecting power" of the healthy carriers of group I—IV may be gauged by the result of experiments 2*a*, 2*b*, 3, 4, 4*a*, 5, 6, 6*a* and 7. These experiments show that such carriers may infect 53—90 pct. of the anopheles which bite them *once*, even if male gametocytes are too scarce to be detected within a reasonable time on the day of infection, so long as these gametocytes can be found at some earlier date (compare the successful experiments 3 and 6*a* with the unsuccessful ones 4, 4*a* and 5 [1])).

In the same way the "infecting power" of the healthy carriers of group V and VI may be assessed by the result of experiments 1 and 2. These experiments show that carriers with very few parasites may still infect anopheles which bite them *once*; but the resulting rate of mosquito-infection is a low one. The provision with regard to male gametocytes is the same as in the preceding paragraph.

Experiments, 2, 2*a* and 2*b* are of special interest. They refer to the same carrier who remained afebrile since January 6. On January 12, the parasite-count being 1 in 1000, he infected 3 anopheles out of 81. On January 19 the parasite-count rose to 15 in 100 and again, on January 22, to 56 in 100. On January 19 this carrier infected anopheles up to 63 pct. His parasite-relapse was not accompanied by a clinical relapse. Consequently, under natural conditions, no physician nor medicines would have interfered with the process of anopheline infection. Provided one and the same batch of anopheles is staying in the carrier's house feeding off and on all the time, some of them are bound to bite the parasite carrier at the moment his parasites are numerous and then they will become heavily infected, just like batch 2 would have done if it had continued to feed on the carrier as long as batches 2*a* and 2*b*.

Elsewhere [2]) we have said that the essential condition for anopheles to become heavily infected by feeding on healthy carriers is to feed repeatedly on him, i.e. to stay with him in the same house for a considerable time. We can now specify this condition by adding that this statement does not mean that a carrier with few parasites cannot infect anopheles at one sitting but that he can do so at several. It means that repeated feedings are required to allow anopheles to avail themselves of the increase of the parasites in the carrier which is due to appear at some time or another. We know this requirement is met by the conditions existing in North-Holland houses in late summer and early autumn.

---

[1]) N⁰. 5 is not unsuccessful, but the rate of anopheline infection is much too low compared with no. 3 and 6*a*. The administration of myosalvarsan has obscured the gametocyte findings. The counterpart of this experiment, not recorded in the list, is the carrier with a parasite-count of 1 in 300, without male gamet., on the day 30 anoph. took his blood. None became infected, although male gamet. were present on earlier dates. But the patient had been given neosalvarsan on several occasions.

[2]) Second international congress for microbiology, Section 5, July 28, 1936.

TABLE 1. Result of experimental infection of anopheles with afebrile parasite-carriers.

Number of experiment and initials of carrier	Date of last fever paroxysm	Last date of finding male gametocytes before anopheles were infected	Anopheline infection					Remarks
			Date of infection	Number of parasites in carrier	Date of dissection	Number of anoph. dissected	Number found infected and number of oocysts	
1. Bl.	April 19	April 15 : 3 ♂ per 100 leuc.	April 27	1 parasite per 6000 leucocytes. No ♂	May 4	42	1 infected 1 oocyst	
2. Br.	Jan. 6	Jan. 4 : 2 ♂ per 100 leuc.	Jan. 12	2 parasites per 2000 leucocytes. No ♂	Jan. 18	81	3 infected 2, 1 and 1 oocysts	
2*a*. Id.	Id.	Id.	Jan. 19	29 parasites per 200 leucocytes. No ♂	Jan. 23 and later	30	19 infected. Average number of oocysts 16	Same carrier as no. 2; had no more fever, notwithstanding rise in parasites
2*b*. Id.	Id.	Id.	Jan. 22	112 parasites per 200 leucocytes. 1 ♂ per 200 leucocytes	Used for infecting patients. Consequently rate of infection not established exactly			
3. P.	Aug. 8	Aug. 10 : 1 ♂ per 200 leuc.	Aug. 12	5 parasites per 500 leucocytes. No ♂	Aug. 17	52	47 infected. Average number of oocysts 6	

TABLE 1. *(Continued).*

4. Hr.	May 3	♂ always absent	May 5	3 parasites per 200 leucocytes. No ♂	May 11	26	None	
4*a*. Id.	June 3	♂ always absent	June 25	3 parasites per 100 leucocytes. No ♂	June 30	45	None	Same carrier as no. 4 after relapse on June 1 and 3
5. Zw.	Jan. 26	♂ always absent	Febr. 18	10 parasites per 200 leucocytes. No ♂	Febr. 25	40	1 infected 1 oocyst	carrier had myosalvarsan injected on Jan. 25 (100 mg.), Febr. 1 (75 mg.) and Febr. 11 (100 mg.)
6. M.	June 10	June 15 : 1 ♂ per 200 leuc.	June 17	18 parasites per 200 leucocytes. No ♂	June 23	10	8 infected. Average number of oocysts 24	
6*a*. Id.	July 7	July 7 : 1 ♂ per 100 leuc.	July 17	18 parasites per 300 leucocytes. 1 ♂ per 3000 leucocytes	July 22	38	26 infected. Average number of oocysts 6	Same carrier as no. 6 after relapse from June 27 till July 7
7. C.	May 13	May 8 : 10 ♂ per 100 leuc.	May 18	51 parasites per 100 leucocytes. 2 ♂ per 100 leucocytes	Used for infecting patients. Consequently rate of infection not established exactly.			

2. *"Healthy" parasite carriers among adults and children.*

31 families comprising 80 adults and 184 children have been examined in spring and early autumn of 1936.

In that year 16 adults and 62 children had malaria 1), an incidence of 20 and 34 pct. respectively. The parasite carriers numbered 24 or 30 pct. among the adults and 79 or 43 pct. among the children.

The 24 adult carriers had more malaria in 1936 than the other adults, viz. 8 of them (33 pct.), whereas 8 only out of 56 non-carriers had malaria (14 pct.).

In children conditions are different: 28 out of 79 carriers had malaria (35 pct.) and 34 out of 105 non-carriers (32 pct.).

Consequently the carriers among the children are not in a worse position than the non-carriers, but the carriers among the adults are.

The following table (2) shows the number of parasites and the presence of male gametocytes in the "healthy" carriers under observation.

TABLE 2. Number parasites and presence of male gametocytes in "healthy" carriers.

	Number of carriers who had had malaria in 1936 and who were carrying the following number of parasites						Number of carriers who had had no malaria in 1936 and who were carrying the following number of parasites					
	1 or more per 100 leucocytes		1—5 per 1000 leucocytes		less than 1 per 1000 leucocytes		1 or more per 100 leucocytes		1—5 per 1000 leucocytes		less than 1 per 1000 leucocytes	
	all	with male gam.	all	with male gam.	all	with male gam.	all	with male gam.	all	with male gam.	all	with male gam.
24 adult-carriers	5	2	1	0	2	0	1	0	4	1	11	1
79 children-carriers	6	3	5	0	17	0	10	3	12	4	29	0

Taking separately the carriers with 1 parasite or more per 100 leucocytes (i.e. the carriers which were shown in section 1 to be able to cause a heavy anopheline infection) and calling them *"good carriers"* we find:

1. 6 good carriers in 62 children who had malaria in 1936 (10 pct.) and 10 good carriers in 122 children who had not (8 pct.), i.e. the good carriers are about equally numerous in both groups.
2. 5 good carriers in 16 adults who had malaria in 1936 (31 pct.) and

---

1) The term "malaria" means malaria fever,

1 good carriers in 64 adults who had not (1½ pct.), i.e. nearly all good carriers are to be found among the adults who had malaria in the course of the year.

3. 3 carriers of male gametocytes in 62 children who had malaria in 1936 (5 pct.) and 7 in 122 children who had not (6 pct.), i.e. the gametocyte carriers are about equally numerous in both groups.

4. 2 carriers of male gametocytes in 16 adults who had malaria in 1936 (12 pct.) and 2 in 64 adults who had not (3 pct.), i.e. the gametocyte carriers are four times more numerous among the adults who had malaria in the course of the year than among those who had not.

Whatever may be the explanation of this difference between adults and children, the broad fact remains that adult carriers likely to cause a heavy anopheline infection, judging by the number of their parasites (see section 1) or by the presence of male gametocytes, are more liable than children to become conspicuous by an attack of malaria.

So far as that goes we may conclude that children carriers are more important as a source of anopheline infection than adult carriers. But there can exist no doubt that adults sometimes are very important in this respect. Last year, in Wormerveer we met with the following case which is well appropriate to illustrate this statement:

In family Kr. the four adult members (there are no children) had malaria: H. on July 30, M. on August 31, W. on Sept. 9, K. on Oct. 8. They were all treated with a 7 days' course of atebrin, 0.3 grammes a day, and had no relapses. Infected anopheles were found on October 8 (17 infected out of 77, 16 sporozoite carriers), November 2 (8 infected out of 25, 6 sporozoite carriers) and December 8 (4 infected out of 11, 3 sporozoite carriers, the majority of the sporozoites degenerated). The source of anopheline infection was M., who had had no more malaria after his atebrin cure, but who nevertheless was found to carry 22 parasites, comprising 1 male gametocyte per 500 leucocytes on October 30. On that same date W. was carrying 3 parasites per 6000 leucocytes, H. and K. had no parasites.

Conclusions.

1. Healthy carriers with one parasite per 100 leucocytes can infect 60 pct., or more, of the anopheles which fed on them only once, even if no male gametocytes could be detected on the day of infection, provided gametocytes had been found at some earlier date and no salvarsan had been given. The same applies to healthy carriers with one parasite per 1000 leucocytes or less, who can likewise infect anopheles, but at a much lower rate.

2. Anopheles sharing a house with a human parasite carrier for a long time, are bound sooner or later to acquire the infection if they continue

to feed. For they are always on the spot to catch the first opportunity offered by a temporary rise in the number of the carriers' parasites, especially if this rise is not accompanied by any marked febrile reaction.

3. Children are in a better position to infect anopheles than adults, because "good carriers" (with 1 parasite per 100 leucocytes or more) are as numerous in children who had no malaria in the course of the year as in those who had. In adults, on the contrary, nearly all "good carriers" are found among those who had malaria in the course of the year and so these adult carriers are more likely to be found out.